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Statement

and

Readings

The Necessity of Quantum Mechanics

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Abstract

The usual formulation of quantum mechanics rests on elaborate mathematical constructs that lack any intuitive grounding. One may therefore wish to find arguments for the rational necessity of this theory. I will sketch three arguments of this kind. The first is the proof, given in 1979 by André Lichnerowicz and Simone Gutt, that every one-parameter continuous deformation of the Poisson algebra of classical mechanics is equivalent to the algebra of infinitesimal evolutions of quantum mechanics. The second is the quantum logic initiated in 1936 by John von Neumann and Garrett Birkhoff, which purports to derive the matrix-density representation of states from a natural logic of Yes-No empirical questions. The third, dating from 2001, is Lucien Hardy's simpler derivation of this representation from "five reasonable axioms" about transition probabilities between discrete measurement outcomes. I will compare the assumptions, deductions, and conclusions of these arguments, and try to estimate the extent to which they establish the necessity of quantum mechanics.

THE NECESSITY OF QUANTUM MECHANICS

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Warning: *This is only a rough draft of a forthcoming study, not ripe to be circulated or cited. The four sections can be read independently. Comments and criticisms are most welcome.*

Quantum mechanics is hard to swallow. It relies on elaborate mathematical constructs whose empirical relevance boggles the mind. Are their ways to make it more natural? In other words: Are there convincing arguments for the necessity of quantum mechanics?

The first section of this essay is devoted to two kinds of historical necessity of quantum mechanics, one driven by the analogy between classical and quantum theory, the other by the analogy between matter and light. The second section is devoted to a mathematical kind of necessity: the possibility of deriving quantum mechanics by deforming the Poisson algebra of classical mechanics. Around 1940, José Moyal and Hibrand Groenewold discovered that quantum mechanics admitted a phase-space formulation that increased its formal similarity with Hamiltonian mechanics: one just had to replace the Poisson bracket by the more complicated "Moyal bracket." In the late 1970s, the mathematicians Jacques Vey, André Lichnerowicz, and Simone Gutt found that the latter bracket could be obtained by deforming the Poisson bracket and that this deformation was unique. This means that in a deep mathematical sense classical mechanics already contains quantum mechanics.

The two remaining sections of this essay deal with attempts to base quantum mechanics on axioms more natural than those of the standard textbook formulation. Since the invention of quantum mechanics, many different axiomatics have been proposed for quantum theory, with various motivations including mathematical rigor and completeness, conceptual and structural clarity, broader scope (for systems resisting standard quantization), deeper interpretive or ontological significance, physical transparency, and necessity. I only retain the axiomatics that have this last motivation. (This explains, for instance, why I do not discuss the C^*-algebraic approach, despite its seductive generality and mathematical power.) Section 3 deals with the so-called "quantum logic," which purports to derive the basic structure of quantum mechanics from simple, natural requirements about the logic of Yes-No empirical questions. In this approach, which originated in John von Neumann's rigorous reformulation of quantum mechanics, the specificity of quantum phenomena is traced to the existence of incompatible questions. The last and fourth section is devoted to a more recent kind of axiomatics, based on natural constraints on a statistical-operational definition of physical states. In this approach, which originated in a seminal paper of 2001 by Lucien Hardy, discreteness, probability, and information are the most important notions. In the past few years, heavy weight has been placed on the information-theoretic meaning of the axioms, sometimes with reductionist intentions.

This essay does not directly address deeper interpretive questions about the nature of the quantum world. It is limited to arguments for the necessity of the basic predictive apparatus of quantum mechanics, although such arguments may suggest new interpretive insights and although some of their authors were motivated by deeper interpretive questions.[1]

1. Historical necessity

I suppose my reader to be somewhat familiar with the history of quantum theory.[2] He or she may still wonder whether the main innovative steps of this history were in some sense necessary. The question is not easy to answer because the implied necessity may belong to different categories. In a first category, the novel elements are deduced from well-defined physico-mathematical principles, in a quasi-rational way. In a second category, the novel elements are induced from well-established empirical data (together with well-confirmed, lower-level theories). In a third category, they may result from the intractability or unavailability of alternative approaches. In a fourth and last category, they may be the resultant of psychological or social factors, implying for instance the authority of a leader. In a philosophical dream-world, the two first categories would be dominant. Needless to say that in the real world the first and second kinds of necessity are often contaminated by the third and fourth. Also, the distinction between deductive and inductive necessity can only be a loose one, because induction usually requires established principles, and because principles often have a partly empirical origin.

An additional difficulty results from the variability of the kind of necessity of a given innovation when a fine time-scale is used. Most frequently, the innovation is initiated by a single actor for reasons that have to do with his personal itinerary, his cultural immersion, and his psychological character. At this early stage, he may be the only one to regard his move as inductively or deductively necessary, while other actors may be skeptical and regard the move as arbitrary. At a later stage a critical debate usually occurs, at the end of which the majority of experts agree that the step must be taken for reasons that may vary from case to case: confirmation of the move by new empirical data, theoretical consolidation of the original deduction, availability of independent deductions that lead to the same result, or compatibility with independent, fruitful developments.[3]

In most of the following discussion, I will judge the necessity of the innovative steps at the end of this second stage. I will not examine the nature of the first stage; for instance, I will not ask whether Bohr's familiarity with Harald Høffding's philosophy or Born's awareness of the anti-causal philosophies of the Weimar period inspired their most daring moves. At any rate, the closest approximations to inductive or deductive necessity are more likely to be found in the justification stage.[4]

The early twentieth-century conclusion that ordinary electrodynamics could not yield equilibrium for thermal radiation comes close to the ideal of deductive necessity.

[1] For a lucid justification of the constructive, axiomatic approach, see Grinbaum 2007.
[2] For this purpose Darrigol 2009 should be sufficient.
[3] These two stages are vaguely similar to Hans Reichenbach's distinction between context of discovery and context of justification.
[4] Social constructivists would agree with me that this second stage is essential in stabilizing the basic constructs of science. However, their analysis of the stabilizing process tends to underestimate rational constrains or to reduce them to socially defined systems of beliefs.

The lack of rigor in the implied deductions was compensated by the multiplicity and variety of derivations of the same result. Moreover, the status of one of these derivations, the Gibbsian one provided by Lorentz, rose with the conviction that Gibbs's ensembles correctly represented thermodynamic equilibrium despite the lack of a firm foundation.

The introduction of quantum discontinuity early in the twentieth century obeyed a weaker necessity of the inductive kind. Einstein's and Bohr's discrete quantization was the simplest way to account for Planck's blackbody law and for the spectrum of the hydrogen atom. Yet it was highly problematic for two reasons: it implied a non-classical selection among classically defined states, and it made it very difficult to imagine a plausible mechanism for the interaction between atoms and radiation. For the latter reason, Planck long preferred a division of phase space into cells of equal a priori probability. As is well known, in 1911 Paul Ehrenfest and Henri Poincaré proved that the canonical distribution of energy over resonators could not yield a finite energy for cavity radiation unless there was a finite energy threshold for the excitation of the resonators.[5] This proof is largely illusory, because it depends on an unwarranted extension of Gibbs's canonical distribution law to systems that no longer obey the laws of classical dynamics.[6] The true reason why Einstein's idea of a sharp quantization came to dominate over more timid attempts was the multiple, successful applications it had in the context of the Bohr-Sommerfeld theory.

Similar comments can be made about Bohr's frequency rule. It is tempting to say that Bohr read the rule in the Balmer-Rydberg formula. In reality, this inference has the typical underdetermination of any inductive reasoning: the hydrogen spectrum can be derived from the combination of the frequency rule only if when this rule is combine with a few other assumptions including the existence of stationary states and the truth of the laws of wave optics for the emitted radiation. Bohr himself did not believe in the generality of the frequency rule until he became aware of Sommerfeld's and Einstein's contributions to his theory in 1916. The assumption of stationary states and the frequency rule gained credibility and became Bohr's two "postulates" when their simultaneous application yielded correct results in an increasing variety of situations involving spectra, atomic structure, and atomic collisions. This happened despite the evident incompleteness of the theory (it left the radiation mechanism in the dark) and despite its reliance on classical concepts belonging to an incompatible electrodynamics.

The very definition of stationary states and the statement of the frequency rule required classical concepts: energy and frequency. Bohr struggled to show that these concepts could be defined in the quantum realm through a limited use of classical theory that did not contradict the quantum postulates. Most important, his correspondence principle pointed to a deep formal analogy between classical electrodynamics and the evolving quantum theory. He hoped that in the long run this analogy would project the consistency of the former theory over the latter. The quantum postulates would remain intact in this process.

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[5] Cf. Klein 1970a; *CD*, p. 53.

[6] In his statistical mechanics, Gibbs assumed the validity of Hamiltonian dynamics. Although Einstein did not in his own statistical mechanics, he still assumed continuous evolution, invariance of the volume element in phase space, and some weak ergodicity. In the 1910s, there already were reasons to doubt the validity of any of these requirements in the case of quantum systems.

Although Bohr reached the correspondence principle by analogy with classical electrodynamics, he insisted on the formal character of this analogy and emphasized the contrast between the quantum postulates and the continuity of classical radiation processes. In order to judge the necessity of this principle, one must first be aware that in Bohr's original view this principle was a relation between the periodicity properties of the motion in stationary states (whether or not this motion obeyed classical mechanics) and the properties of the emitted radiation. There were three arguments in favor of the necessity of this principle: it warranted the asymptotic agreement between the empirical predictions of classical and quantum theory; in the deductive mode, it provided the selection rules and good estimates of the intensities of some spectral lines; through Bohr's more obscure appeal to the inductive mode, it led to a plausible classification of elements.

The magic of the correspondence principle did not catch well outside Copenhagen. By 1924, the idea of well-defined orbits in the atom, which the principle seemed to require, was much under criticism. Even Bohr came to reject this idea in early 1925. Yet in Bohr's circle the confidence never died that correct quantum-theoretical relations could be extracted by analogy with classical multiperiodic systems, whether or not the motion of such systems truly represented the motion in stationary states. This confidence even increased in 1923-24 when Kramers, Born, and Heisenberg managed to translate some classical relations into what they (correctly!) believed to be exact quantum-mechanical relations. One reason for this belief was the empirical relevance of these relations. Another was the automatic agreement between the large-quantum-number limit of these relations and the corresponding classical relations. Still another was the fact, first emphasized by Kramers, that these relations only involved the basic quantities entering Bohr's postulates and no longer referred to the suspicious orbits. In the spring of 1925, Heisenberg's conviction that he had discovered quantum mechanics resulted from these three qualities of the symbolic translation, together with the consistency and completeness of the resulting computational scheme.

Heisenberg's quantum mechanics may be regarded as a necessary consequence of Bohr's two postulates (discrete stationary states, and frequency rule) and of a rule for translating the equations of motion of a classical periodic system (expressed in Fourier form) into relations between "quantum amplitudes" directly related to the observable quantities that enter the two quantum postulates. This rule itself derived from the correspondence principle, whose plausibility rested on the asymptotic validity of classical electrodynamics and on successful applications (of a different kind) in the earlier quantum theory. One might then wonder why quantum mechanics was not discovered earlier, say in 1917, when Bohr already had the two postulates as the pillars of his theory and the correspondence principle as a constructive tool. One reason is that before 1924 no one banished orbital parameters from quantum theory. Another is that before Heisenberg no one guessed that the "correspondence" counterparts of the Fourier components of a periodic classical motion would completely characterize the quantum-mechanical motion just as these components themselves sufficed to define the classical motion.

On the side of wave mechanics, the story began with de Broglie's extension of the wave-particle duality to particles of finite mass. Although the extension was natural from a formal, relativistic point of view, it could easily pass for crazy speculation. The receptivity of Langevin, Einstein, and Schrödinger depended on a few favorable circumstances. Firstly, the lightquantum, which provided the basis for de Broglie's extension, was gaining momentum (literally and metaphorically). Secondly, de Broglie's successfully applied his theory to a wide spectrum of problems including the derivation of the Bohr-Sommerfeld rule, the analogy between Fermat's and Maupertuis's principles, and the derivation of Planck's quantum cells (for the statistics of gas molecules). Thirdly, Einstein retrieved the de Broglie waves through a different route: in 1925 he designed a quantum theory of gas degeneracy by analogy with Bose's corpuscular derivation of Planck's law, and found that the theoretical fluctuation of his quantum gas implied wave behavior in conformity with de Broglie's relations. Being also involved in quantum-gas theory, Schrödinger measured the force of Einstein's reasoning.[7]

It would nonetheless be excessive to speak of a deductive or inductive necessity of de Broglie's waves. In 1925, they still were a bold assumption without direct experimental counterpart.[8] De Broglie was himself shy in his suggestion of electron diffraction.[9] A stronger necessity can be seen in the deduction of the Schrödinger equation. De Broglie's idea that the classical dynamics of a particle should be to wave mechanics what geometrical optics is to wave optics automatically leads to the time-independent Schrödinger equation in the non-relativistic limit. Moreover, the success of this equation in determining the stationary states of the hydrogen atom could hardly be regarded as pure chance. One may still be perplexed by the coincidence that made the Schrödinger equation appear just a few months after Heisenberg's quantum mechanics. There is little to justify this timing besides the contemporary willingness to renounce electronic orbits in atoms.

A last question of special philosophical interest is the necessity of the now standard probabilistic interpretation of the formalism of quantum mechanics or wave mechanics. In Dirac's quantum-mechanical approach, the starting point is the allegation that for a sharply defined value of the energy (corresponding to a stationary state) the conjugate phase is uniformly spread. The ensuing deduction of the whole interpretation only requires the transformation properties of the fundamental equations of quantum mechanics (invariance by unitary transformations). The necessity of this interpretation should therefore be measured by the necessity of the starting point. Dirac justified his starting point through the correspondence principle, arguing that in the large quantum-number limit a stationary state may be represented by a revolving electron whose phase varies uniformly in time. Thus, Dirac was willing to admit that energy and phase retained a meaning in quantum mechanics. More generally, he assumed that any dynamical variable and its canonical conjugate retained a meaning in quantum mechanics although it was impossible to have initial conditions in which both variables were exactly determined. There is no evident necessity for this persisting relevance of classical concepts in the quantum context. Nevertheless, the harmony of Dirac's statistical interpretation with the transformation properties of quantum mechanics pleaded for the uniqueness of this interpretation.

In the matter-wave approach, Born's probabilistic interpretation of scattered electron waves seems unavoidable. Indeed the naïve interpretation of the wave as dilute matter would imply that only a fraction of an electron is detected at a given angle. Any

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[7] Einstein 1924, 1925a, 1925b. Cf. *CD*, pp. 248-249; *HD*, vol. 1, chap. 5.3; Forman 1969; Hanle 1977.
[8] Walther Elsasser nonetheless tried to relate de Broglie waves to anomalies observed in Göttingen for the scattering of low-energy electrons. Cf. Born 1926b; *CD*, pp. 249-251; Russo 1981.
[9] The suggestion only appears in Broglie 1923b, p. 549, not in the *Thèse* (Broglie 1924).

attempt to save the naïve view by building wave packets of very small size would fail because of the spreading of the wave packets. Similarly but less stringently, Dirac's statistical interpretation of the perturbed Schrödinger-wave of an irradiated atom seems to result from the nature of the problem, granted that long after the interaction the atom can only be found in a stationary state. Lastly, it is possible to show that Born's probabilistic interpretation of scattered waves leads to the full statistical interpretation of wave mechanics through a proper idealization of the measuring process [proof omitted]. One is left with a feeling of the unavoidability of the standard statistical interpretation of wave or matrix mechanics. Its ability to correctly represent the outcome of experiments in the quantum regime has rarely been contested. The apple of later discord rather was the possibility of defining or measuring physical quantities more than quantum mechanics allows.

To sum up, the historical genesis of quantum mechanics can be regarded as a series of bold, imaginative, but firmly supported steps. The first two constructive steps, the introduction of discrete stationary states and the frequency rule, were taken with full awareness of their problematic character and they were later consolidated by multiple successes of their combined application. These assumptions have counterparts in modern quantum mechanics, although stationary states are no longer regarded as the only possible states. In contrast, the auxiliary reliance on classical concepts posed more and more problems and led to the severe crisis of 1924-25. In the middle of this crisis, Heisenberg's invention of a first form of quantum mechanics strikingly confirmed Bohr's idea that the correspondence principle promised a "rational generalization" of classical electrodynamics. The contemporary but largely independent invention of a closely related wave mechanics strengthens the air of inevitability of quantum mechanics. The statistical interpretation of this theory largely derives from its mathematical structure, combined with a touch of correspondence.

Rational arguments from the history of quantum mechanics of course lack rigor and purity. They rely on arbitrary idealizations and they are often contaminated by appeal to experimental knowledge. These arguments nevertheless suggest natural assumptions from which quantum mechanics might follow deductively.

2. The deformation of classical mechanics

One suggestion we can take from history is that quantum mechanics should be some sort of "rational generalization" (as Bohr put it) of classical mechanics. The standard recipe for building a quantum mechanical Hamiltonian, namely, replacing canonical pairs with non-commuting operators, does not meet anyone's spontaneous notion of rationality. Nevertheless, Weyl and Wigner independently discovered reciprocal, unambiguous ways to associate an operators with functions in ordinary phase space. Some fifteen years later, Moyal and Groenewold used this correspondence to reformulate quantum mechanics in phase space. In particular they translated the products of two operators into the "Moyal product" of the corresponding functions in phase space, and the commutator of two operators into a generalization of the Poisson bracket of two functions in phase space. In the 1970s, mathematical studies of deformations of the classical Poisson algebra of infinitesimal evolutions done by André Lichnerowicz's group, Jacques Vey, and Simone

Gutt led to a proof that the Moyal bracket was the unique continuous deformation of the Poisson bracket. This section is a historico-critical analysis of this astonishing discovery, beginning with the earliest conceptions of quantization and ending with comments on the sort of necessity it implies for quantum mechanics.

Early connections between classical and quantum mechanics

The history of quantum mechanics, the correspondence principle, and the Bohrian interpretation of this theory in terms of classical observation all suggest an intimate connection between classical and quantum mechanics. In the simple case for which the classical Hamiltonian function has the form

$$H(p,q)=\frac{p^2}{2m}+V(q),$$

the quantum-mechanical Hamiltonian has the same form

$$\mathbf{H}=\frac{\mathbf{p}^2}{2m}+V(\mathbf{q}),$$

and the equation of motions in the Heisenberg picture still are Hamilton's equations

$$\dot{\mathbf{q}}=\frac{\partial\mathbf{H}}{\partial\mathbf{p}}, \quad \dot{\mathbf{p}}=-\frac{\partial\mathbf{H}}{\partial\mathbf{q}}.$$

The only formal difference hinges on the commutation rule

$$[\mathbf{q},\mathbf{p}]=i\hbar.$$

In the Schrödinger picture, the Schrödinger equation

$$i\hbar\frac{\partial\psi(q,t)}{\partial t}=H\left(q,-i\hbar\frac{\partial}{\partial q}\right)\psi(q,t)$$

is the only first-order wave equation whose eikonal approximation leads to the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t}=H\left(q,\frac{\partial S}{\partial q}\right),$$

for the phase S of the waves.

There is much boldness in either way of generating the quantum-mechanical. One way introduces non-commuting quantities; the other turns particles into waves. No classical physicist would have taken the resulting equations seriously. Yet the formal-mathematical kinship between classical and quantum mechanics goes even deeper than suggested by their historical constructions. In 1925, Dirac noted that Heisenberg's equations, once rewritten under the form

$$[\mathbf{q},\mathbf{p}]=i\hbar, \quad \dot{\mathbf{g}}=(i/\hbar)[\mathbf{H},\mathbf{g}],$$

were the exact counterpart of the classical equations

$$\{q,p\}=1, \ \dot{g}=-\{H,g\},$$

in which the Poisson bracket of two functions f and g of q and p is defined as

$$\{f,g\}=\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}.$$

That is to say, the quantum-mechanical equations can be obtained through the simple correspondence

$$\{\ ,\ \} \rightarrow (i\hbar)^{-1}[\ ,\ ].$$

In mathematical terms, the Poisson algebra of classical Hamiltonian infinitesimal evolutions seems to be mapped into the Poisson algebra of quantum Hamiltonian infinitesimal evolutions. In his earliest work on quantum mechanics, Dirac emphasized this correspondence and used it abundantly to adapt classical methods of resolution to the quantum domain.[10]

Unfortunately, the correspondence between the two theories is not as simple as Dirac wished. In December 1925, Heisenberg warned Dirac that the alleged correspondence between Poisson brackets and commutators could not hold for every quantity. For instance, if $[\mathbf{q},\mathbf{p}]$ matches $i\hbar\{q,p\}$ then $[\mathbf{q}^2,\mathbf{p}^2]$ cannot match $i\hbar\{q^2,p^2\}$ because

$$[\mathbf{q}^2,\mathbf{p}^2] = \mathbf{q}[\mathbf{q},\mathbf{p}^2] + [\mathbf{q},\mathbf{p}^2]\mathbf{q} = \mathbf{qp}[\mathbf{q},\mathbf{p}] + \mathbf{q}[\mathbf{q},\mathbf{p}]\mathbf{p} + \mathbf{p}[\mathbf{q},\mathbf{p}]\mathbf{q} + [\mathbf{q},\mathbf{p}]\mathbf{pq} = 2\hbar(\mathbf{qp}+\mathbf{pq})$$

whereas[11] $\{q^2,p^2\} = 4qp$.

Weyl and Wigner

This counterexample brings forth a more general difficulty in defining the quantum-mechanical operator **g** corresponding to a given classical quantity $g(q,p)$. Whenever the power series-development of the *g* function involves terms of the form $p^m q^n$, there is an ambiguity in the quantum translation. For instance, should pq translate into $\mathbf{pq}$, $\mathbf{qp}$, or $\frac{1}{2}(\mathbf{pq}+\mathbf{qp})$? In 1927, Hermann Weyl approached this question from a group-theoretical point of view. In this context, the Hermitic operators **q** and **p**, such that $[\mathbf{q},\mathbf{p}] = \mathrm{i}\hbar$, are the generators of the Lie algebra of the possible unitary evolutions of the quantum system. More exactly, any such evolution can be obtained by taking the exponential of a linear combination $\mathrm{i}(\alpha\mathbf{q}+\beta\mathbf{p})$ with real coefficients α and β. Moreover, any Hermitic operator **g** of the Lie algebra can be obtained as a superposition

$$\mathbf{g} = \int \tilde{g}(\alpha,\beta)\mathrm{e}^{\mathrm{i}(\alpha\mathbf{q}+\beta\mathbf{p})}\mathrm{d}\alpha\mathrm{d}\beta,$$

with $\tilde{g}^*(\alpha,\beta) = \tilde{g}(-\alpha,-\beta)$.

Weyl introduced this decomposition in analogy with Fourier analysis, and because he wanted to circumvent the unboundedness of the operators **p** and **q** of quantum mechanics (which is an obstacle to its Hilbert-space representation).[12]

The condition $\tilde{g}^*(\alpha,\beta) = \tilde{g}(-\alpha,-\beta)$ implies that the ordinary Fourier transform of the coefficients $\tilde{g}(\alpha,\beta)$,

$$g(q,p) = \int \tilde{g}(\alpha,\beta)\mathrm{e}^{\mathrm{i}(\alpha q+\beta p)}\mathrm{d}\alpha\mathrm{d}\beta$$

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[10] Dirac 1925.
[11] Ref. in CQ
[12] Weyl 1927, pp. 27-28.

is a real function of the real variables q and p. Weyl naturally interpreted this function as the classical quantity corresponding to the quantum quantity **g**. He thus had in hand a unique, group-theoretically sound way of associating a quantum operator to a classical quantity. In compact form, the recipe reads:

$$\mathbf{g} = \frac{1}{4\pi^2}\int g(q,p)\mathrm{e}^{\mathrm{i}[\alpha(\mathbf{q}-q)+\beta(\mathbf{p}-p)]}\mathrm{d}q\mathrm{d}p\mathrm{d}\alpha\mathrm{d}\beta .$$

This rule implies the complete symetrization of monomes. For instance,[13]

$$qp \to \frac{1}{2}(\mathbf{qp}+\mathbf{pq}), \qquad\qquad q^2 p \to \frac{1}{3}(\mathbf{q}^2\mathbf{p}+\mathbf{qpq}+\mathbf{pq}^2).$$

Five years later, Eugen Wigner addressed the seemingly unrelated question of the representation of quantum states in phase space. His aim was to devise a quasi-classical approximation strategy for the quantum statistical averages

$$\langle \mathbf{g}\rangle = \mathrm{Tr}(\boldsymbol{\rho}\mathbf{g}), \quad \text{with } \boldsymbol{\rho} = \mathrm{e}^{-\beta\mathbf{H}}/\mathrm{Tr}(\mathrm{e}^{-\beta\mathbf{H}}).$$

In order to ease the comparison, with the classical statistical average[14]

$$\bar{g} = \int \rho(q,p)g(q,p)\mathrm{d}q\mathrm{d}p ,$$

he associated a phase-space distribution $\rho(q,p)$ to any density operator $\boldsymbol{\rho}$ according to the formula

$$\rho(q,p) = 2\int \mathrm{d}q'\,\mathrm{e}^{2\mathrm{i}pq'/\hbar}\langle q-q'|\boldsymbol{\rho}|q+q'\rangle ,$$

and showed that for any quantity **g** that is a sum of a function of **p** and a function of **q**, the quantum average could be replaced with a phase-space average over this distribution:[15]

$$\mathrm{Tr}(\boldsymbol{\rho}\mathbf{g}) = \int \rho(q,p)g(q,p)\mathrm{d}q\mathrm{d}p .$$

In the case of a pure state $\boldsymbol{\rho} = |\psi\rangle\langle\psi|$, the associated phase-space distribution reads[16]

$$\rho(q,p) = 2\int \mathrm{d}q'\,\mathrm{e}^{2\mathrm{i}pq'/\hbar}\psi^*(q+q')\psi(q-q').$$

Wigner did not tell how he had arrived at this miraculous formula. He only indicated that Leo Szilard and himself had obtained it a few years later in another context. Possibly, the two friends had been looking for a phase-space formulation of quantum mechanics, in an attempt to reduce quantum-mechanical probabilities to ordinary probabilities. Wigner's paper indeed contains the phase-space counterpart of the Schrödinger equation:

$$\frac{\partial\rho}{\partial t} = -\frac{p}{m}\frac{\partial\rho}{\partial q} + \sum_{n=0}^{+\infty}\frac{(-1)^n\hbar^{2n}}{(2n+1)!}\frac{\partial^{2n+1}V}{\partial q^{2n+1}}\frac{\partial^{2n+1}\rho}{\partial p^{2n+1}},$$

when $\mathbf{H} = \mathbf{p}^2/2m + V(\mathbf{q})$.

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[13] Weyl (1927, pp. 17) gave the combination rules for the functions $\tilde{f}$ and $\tilde{g}$. So did too Neumann (1931), who used the Weyl-Fourier analysis of Hermitic operators in a proof of the uniqueness of the Schrödinger representation of the operators **p** and **q**.

[14] For the sake of simplicity, I only give the formulas in the case of one degree of freedom.

[15] As we will see in a moment, this property in fact holds for any operator **g**.

[16] Wigner 1932, pp. 750, 753.

This is the quantum-theoretical generalization of the classical equation of evolution of a density in phase-space:

$$\frac{\partial\rho}{\partial t} = -\frac{p}{m}\frac{\partial\rho}{\partial q} + \frac{\partial V}{\partial q}\frac{\partial\rho}{\partial p}$$

Wigner did not fail to note that the quantum-theoretical density ρ, unlike its classical approximation, could take negative values and therefore could not be interpreted as a true probability:[17]

> Of course [the density $\rho(q, p)$] cannot be really interpreted as the simultaneous probability for coordinates and momenta, as it is clear from the fact that it may take negative values. But of course this must not hinder the use of it in calculations as an auxiliary function which obeys many relations we would expect from such a probability.

Moyal, Dirac, and Groenewold

On the one hand, Weyl ascribed an operator to any function in phase-space. On the other, Wigner ascribed a function in phase-space to any operator. Neither of them realized that the implied correspondences were the inverse of each other, presumably because the context and the relevant kind of operator differed. Whereas Weyl wanted to construct the Hermitic operators of quantum mechanics from functions in phase-space, Wigner sought a phase-space representation of the density operators or wave functions of quantum mechanics. Some ten years elapsed before two marginal theorists, José Moyal and Hilbrand Groenewold, discovered the Weyl-Wigner connection in systematic explorations of the phase-space representation of quantum mechanics.[18]

Around 1940, the electrical engineer Moyal, discovered that the average of any operator could be represented as an ordinary average in phase space, if only the **q**'s and **p**'s were properly ordered in the expression of the operator. He took this as an indication that joint probability distributions in q and p could adequately represent quantum evolution, despite the non-positive character of these distributions. Paul Dirac opposed this idea and blocked its publication. In contrast, the statistician Maurice Bartlett supported Moyal and helped him develop his theory. In 1944, Moyal approached Dirac a second time. The reaction was again negative. Dirac argued that phase-space averages generally failed to represent quantum averages, because for instance the phase-space average of pq must be the same as the average of qp, whereas the quantum averages of **pq** and **qp** differ by $i\hbar$. As Moyal explained in his reply, Dirac had misread him, because in Moyal's theory the two kinds of average only agree if the quantum operator is ordered according to Weyl's prescription. Thus, in Dirac's example, the theory only requires that the phase-space average of pq should be equal to the quantum average of $\frac{1}{2}(\mathbf{pq}+\mathbf{qp})$.[19]

[17] Ibid., pp. 750n, 751.

[18] On this history, cf. Curtwright and Zachos 2011.

[19] Moyal to Dirac, 18 Feb. 1944; Dirac to Moyal, 20 Apr. 1945; Moyal to Dirac, 29 Apr. 1945. Cf. Ana Moyal 2006, chaps. 1-3 (biography), appendix 2 (Moyal-Dirac correspondence).

Dirac then raised a more serious objection. He noted that in the case of a harmonic oscillator characterized by the Hamiltonian $H = p^2 + q^2$, Moyal's theory implied a non-vanishing quadratic energy fluctuation in any eigenstate, against the standard view that the energy is sharply defined in such states. Indeed, the Weyl quantization of H^2 and the commutation rule $\mathbf{qp} - \mathbf{pq} = i\hbar$ together lead to[20]

$$(p^2 + q^2)^2 \rightarrow (\mathbf{p}^2 + \mathbf{q}^2)^2 + \hbar^2 / 4 \ ,$$

so that

$$\overline{H^2} - \overline{H}^2 = \left\langle \mathbf{H}^2 \right\rangle - \left\langle \mathbf{H} \right\rangle^2 + \hbar^2 / 4 = \hbar^2 / 4 \, .$$

The source of this paradox is the naïve idea that the quantum-theoretical expectation value of any physical quantity is obtained by Weyl-quantizing its classical expression $g(q, p)$ and forming $\langle \mathbf{g} \rangle = \mathrm{Tr}(\boldsymbol{\rho}\mathbf{g})$. This prescription, or the equivalent identification of the Wigner phase average $\overline{g}$ with the expectation value of *g*, is generally incompatible with the postulate of sharply defined values of a quantity in its eigenstates. This incompatibility, of which Dirac's paradox is an illustration, can be avoided by interpreting the quantum expression $\langle \mathbf{g} \rangle$ as the expectation value of the classical quantity obtained in replacing the **q** and **p** in the expression of **g** by ordinary numbers.[21]

Moyal did not consider this subterfuge. He accepted Dirac's criticism, and published his theory in 1949 with a warning that its statistical prediction did not entirely agree with those of standard quantum mechanics. He even suggested experimental tests of the differences. In 1946, the Dutch theoretical physicist Hilbrand Groenewold, who did not have Dirac on his way, had already published a systematic study of hidden-variable theories which also contained a phase-space representation of quantum mechanics. Whereas Moyal interpreted this representation as a plausible, if imperfect, statistic-deterministic interpretation of this theory, Groenewold used it as a means to argue the impossibility of such an interpretation. Both theorists nevertheless obtained very nearly the same formal apparatus. They were both aware of Weyl quantization, and Groenewold knew about the Wigner function. Bartlett and Moyal obtained this function by inverting the Weyl quantization.[22]

Bartlett became aware of Wigner's contribution in the summer of 1945. Moyal immediately told Dirac that his brother in law had anticipated the phase-space representation. Dirac remained unimpressed. To his earlier objections, he added that the phase-space representation of quantum mechanics, unlike that of classical mechanics depended on the choice of the canonical pair (q, p). He commented:

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[20] Dirac to Moyal, 11 and 18 May 1945.

[21] Under this prescription, the expectation value of the quantity *g* is the phase average of the quantity g', obtained by replacing all ordinary products by star products (defined below, p.) in the polynomial expression of *g*. For instance, the expectation value of H^2 is the phase average of $H' = H * H$, so that the fluctuation in an energy eigenstate is $\overline{H * H} - \overline{H}^2 = \left\langle \mathbf{H}^2 \right\rangle - \left\langle \mathbf{H} \right\rangle^2 = 0$. Cf. Zachos, Fairlie, and Curtright 2005, pp. 17-18.

[22] Moyal to Dirac, 15 and 26 May 1945; Moyal 1949; Groenewold 1946.

If you depart so much from the usual classical ideas is there any point in trying to fit things into a classical framework? What advantages does your system have over the usual statistical interpretation of quantum mechanics? Any results that you get from your system must either conform to the usual quantum mechanics or else be incorrect. I think your kind of work would be valuable only if you can put it in a very neat form.

In the eyes of modern adepts of the phase-space representation, the formal apparatus of Moyal's published memoir is clear and elegant, and its notation is more transparent than Groenewold's. This apparatus will now be given in modern notation.[23]

Quantum mechanics in phase space

If $\tilde{g}(\alpha,\beta)$ denotes the inverse Fourier transform of the classical quantity $g(q,p)$, the Weyl quantization formula implies

$$\langle q'|\mathbf{g}|q''\rangle = \int \tilde{g}(\alpha,\beta)\langle q'|\mathrm{e}^{\mathrm{i}(\alpha\mathbf{q}+\beta\mathbf{p})}|\mathrm{q}''\rangle \mathrm{d}\alpha\mathrm{d}\beta\,.$$

Using the Weyl identity[24]

$$\mathrm{e}^{\mathrm{i}(\alpha\mathbf{q}+\beta\mathbf{p})} = \mathrm{e}^{\mathrm{i}\alpha\mathbf{q}}\mathrm{e}^{\mathrm{i}\beta\mathbf{p}}\mathrm{e}^{-(1/2)[\mathrm{i}\alpha\mathbf{q},\mathrm{i}\beta\mathbf{p}]} = \mathrm{e}^{\mathrm{i}\alpha\mathbf{q}}\mathrm{e}^{\mathrm{i}\beta\mathbf{p}}\mathrm{e}^{\mathrm{i}(\hbar/2)\alpha\beta},$$

we have

$$\langle q'|\mathrm{e}^{\mathrm{i}(\alpha\mathbf{q}+\beta\mathbf{p})}|\mathrm{q}''\rangle = \mathrm{e}^{\mathrm{i}(\hbar/2)\alpha\beta}\mathrm{e}^{\mathrm{i}\alpha q'}\delta(\beta+q'/\hbar-q''/\hbar),$$

and

$$\langle q'|\mathbf{g}|q''\rangle = \int \tilde{g}(\alpha,q''/\hbar-q'/\hbar)\mathrm{e}^{\mathrm{i}(\alpha/2)(q'+q'')}\mathrm{d}\alpha = \frac{1}{h}\int \mathrm{d}p\mathrm{e}^{\mathrm{i}p(q'-q'')/\hbar}g\left(\frac{q'+q''}{2},p\right).$$

Changing the variables to $q=(q'+q'')/2$ and $q_-=(q''-q')/2$, and inverting the Fourier transform yields the Wigner formula

$$g(q,p) = 2\int \mathrm{d}q_-\mathrm{e}^{2\mathrm{i}pq_-/\hbar}\langle q-q_-|\mathbf{g}|q+q_-\rangle\,.$$

This means that the Wigner formula it the exact inverse of the Weyl quantization formula. There is a one-to-one correspondence between distributions in phase-space and quantum operators.

As Moyal and Groenewold both realized, the Weyl-Wigner correspondence is not unique. However, it enjoys two important properties: it yields real (though non-positive) values for the phase-space distributions g associated to Hermitic operators (physical quantities or density matrices); and it translates quantum averages into ordinary phase-space averages:

$$\mathrm{Tr}(\boldsymbol{\rho}\mathbf{g}) = \int \rho(q,p)g(q,p)\mathrm{d}q\mathrm{d}p\,.$$

[23] Moyal to Dirac, 21 Aug. 1945; Dirac to Moyal, 31 Oct. 1945.
[24] This idendity is often called the Glauber identity, even though it plays an important role in Weyl 1927, p. 27.

More generally, for any two real functions $f(q,p)$ and $g(q,p)$ and for the associated Weyl operators **f** and **g**,

$$\mathrm{Tr}(\mathbf{fg}) = \int f(q,p)g(q,p)\mathrm{d}q\mathrm{d}p\,.$$

[proof omitted].

In order to translate quantum-mechanical equations into phase-space equations, it is convenient to introduce the star product $f * g$ whose associated operator is the operator **fg**. In the symbolic notation used by Groenewald and Moyal, This definition yields

$$f * g = f \mathrm{e}^{(\mathrm{i}\hbar/2)(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q)} g\,,$$

in which the arrows above the derivatives indicate on which side they are operating [proof omitted].[25]

To first order in $\hbar$, this formula yields

$$f * g \approx fg + (i\hbar/2)(\partial_q f\, \partial_p g - \partial_p g\, \partial_q f) = fg + (i\hbar/2)\{f,g\}.$$

For the skewsymmetric part, which is the phase-space counterpart of the commutator $[\mathbf{f},\mathbf{g}]$, we have

$$f * g - g * f \approx \mathrm{i}\hbar\{f,g\}\,.$$

The quantum-mechanical equation of motion,

$$\mathrm{i}\hbar\dot{\boldsymbol{\rho}} = [\mathbf{H},\boldsymbol{\rho}],$$

translates into

$$\dot{\rho} = \{\{H,\rho\}\}\,,$$

wherein the Moyal bracket is defined by

$$f * g - g * f = \mathrm{i}\hbar\,\{\{f,g\}\}\,.$$

When $\hbar$ reaches zero, the Moyal bracket reduces to the Poisson brackets and the classical of motion turns into the classical equation of motion $\dot{\rho} = \{H,\rho\}$.

These considerations give a precise meaning to the correspondence between commutators and Poisson brackets. Quantum and classical mechanics now being both represented in phase space, it becomes clear that the Lie algebra of quantum evolutions is a continuous deformation of the Lie algebra of classical evolutions. As Wigner anticipated, the semi-classical limit of quantum mechanics is better understood. The quantum nature of processes can be appreciated by identifying the (small) domains in which the phase-space density is negative. This is why the Wigner function is so popular among physicists who try to understand the transition from quantum to classical behavior, for instance the decoherence process and Schrödinger cats.[26]

Equivalence of all deformations of the Poisson brackets

The existence of the Moyal bracket implies that the basic structure of quantum mechanics can be obtained by deforming the Lie algebra of Hamiltonian mechanics. This result confirms Dirac's intuition that the two theories are not so remote from each other, at least

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[25] Groenewald 1946, pp. 451-452.
[26] Cf. Haroche and Raimond 2006.

from a mathematical point of view. A few mathematicians established a much stronger connection in the 1970s.

The relevant concept of deformation is the one introduced by Murray Gerstenhaber in 1964. Accordingly, a deformed Poisson bracket is a C^{∞} bilinear alternate function $\{f,g\}_{\hbar}$ of the phase-space functions *f* and *g* and of the parameter $\hbar$ defining a Lie Algebra (satisfying the Jacobi identity) in the space of phase-space functions and such that $\{f,g\}_0$ is the usual Poisson bracket. In 1874, André Lichnerowicz and his collaborators Moshé Flato and Daniel Sternheimer discovered non-trivial first-order differential deformations of the Poisson bracket on curved symplectic manifolds. All such deformations are trivial in the flat $\mathbf{R}^{2n}$ case which is the most commonly encountered in Physics.[27]

The following year, Jacques Vey proved the existence of non-trivial deformations of infinite differential order always existed when the third Betti number of the manifold vanished. In the flat $\mathbf{R}^{2n}$ case, Vey's deformed bracket is identical to the Moyal blacket. The Lichnerowicz group soon noted this coincidence. Following a suggestion by Flato, they conceived a new concept of quantization based on deforming the Poisson algebra:

> These developments encourage attempts to view quantum mechanics as a theory of functions or distributions on phase space, with deformed products and brackets. *We suggest that quantization be understood as a deformation of the structure of the algebra of classical observables, rather than as a radical change in the nature of the observables.*

In particular, Flato, Lichnerowicz, and Sternheimer hoped to help theoretical physicists quantizing the constrained Hamiltonian systems that occur in some quantum field theories. Most interestingly, in 1979 Lichnerowicz and Simone Gutt established that all deformations of the Poisson bracket were mutually equivalent on symplectic manifolds for which the second Betti number vanishes. In the flat $\mathbf{R}^{2n}$ case, which is most commonly encountered in physics, all deformations are equivalent to the Moyal bracket.[28]

The equivalence of two deformed brackets $\{f,g\}_{\hbar}$ and $\{f,g\}'_{\hbar}$ is here defined as the existence of a differential operator

$$T = \mathrm{Id} + \sum_{s=1}^{\infty} \hbar^s T_s$$

such that

$$T\{f,g\}'_{\hbar} = \{Tf, Tg\}_{\hbar}$$

for any pair *f, g* of phase-functions. This equivalence evidently preserves the equation of motion $\dot{\rho} = \{H, \rho\}_{\hbar}$. In addition, the equivalence

$$T(f *' g) = Tf * Tg$$

[27] Flato, Lichnerowicz, and Sternheimer 1975.

[28] Vey 1975; Bayen, Flato *et al.* 1978a, p. 62; Lichnerowicz 1979, 1982; Gutt 1979. In the general case, all deformed brackets are equivalent to a Vey bracket, namely a bracket that matches the Moyal bracket in infinitesimal neighborhoods of the manifold.

for associated star product preserves the possibility of translating this equation in operator language. Indeed if we replace the Weyl transform

$$\mathbf{g} = W(g) = \int \tilde{g}(\alpha,\beta) \mathrm{e}^{\mathrm{i}(\alpha\mathbf{q}+\beta\mathbf{p})} \mathrm{d}\alpha \mathrm{d}\beta$$

with the transform

$$W'(g) = W(Tg) = \int \tilde{g}(\alpha,\beta) \mathrm{e}^{\mathrm{i}(\alpha\mathbf{q}+\beta\mathbf{p})} \tilde{T}(\alpha,\beta) \mathrm{d}\alpha \mathrm{d}\beta \,,$$

we have

$$W'(f)W'(g) = W(Tf)W(Tg) = W(Tf * Tg) = W'[T^{-1}(Tf * Tg)] = W'(f *' g) \,.$$

Thus, the *T*-equivalence amounts to an alternative ordering of the quantum operators.[29]

Lichnerowicz's and Gutt's proofs of the equivalence of all star products or of all deformations of the Poisson bracket when the second Betti number vanishes are based on a powerful theorem by Vey regarding the Hochschild cohomology induced by the star product or regarding the Chevalley cohomology induced by the bracket. This result having important bearing on the question of the necessity of quantum mechanics, I will give an elementary proof in the case of a single degree of freedom (one q and one p).[30]
[proof omitted]

Deformation and necessity

The equivalence of all deformations of the Poisson bracket is a highly remarkable result, for it conveys to quantum mechanics a deep mathematical necessity: the generating algebra of this theory is, up to an isomorphism, the only possible deformation of the Lie algebra of the infinitesimal evolutions of Hamiltonian mechanics. So to say, quantum mechanics is implicitly contained in classical mechanics. Recent results by Maurice de Gosson confirm this genetic relationship. In particular, Heisenberg's uncertainty relation can be given a meaning in classical mechanics: there exists a phase-space distribution for which this relation is compatible with the Hamiltonian flow.[31] A more spectacular result concerns the relation between the group of Hamiltonian evolutions in phase-space and the group of unitary evolutions through Schrödinger's equations. For quadratic Hamiltonians, mathematicians have known for some time that the covering group of the former group is identical with the latter. In 2011 Gosson and Basil Hiley proved that for an arbitrary Hamiltonian there still is a one-to-one correspondence between the two kinds of evolution:[32]

[29] The modified Wigner transforms do not preserve the identity of phase-space and quantum averages since $\mathrm{Tr}[W'(f)W'(g)] = \mathrm{Tr}[W(Tf)W(Tg)] = \int TfTg \mathrm{d}p\mathrm{d}q \neq \int fg \mathrm{d}p\mathrm{d}q$]. However, $\mathrm{Tr}[W'(f)W'(g)] = \int f *' g \mathrm{d}p\mathrm{d}q$ remains true, so that the interpretive prescription of note xx above extends to every star product.

[30] This proof has some similarity with the general proof found in Gutt and Rawnsley 1998.

[31] Gosson and Hiley 2009. In the two-dimensional case, this distribution is the uniform distribution within an ellipse of surface h; the $2n$ requires Gosson's subtler notion of "symplectic capacity."

[32] Gosson and Hiley 2011, p. 1434.

> In this way we have shown that the mathematical formalism of the theory of Schrödinger's equation is already present in classical mechanics, and is in fact a reformulation of Hamiltonian dynamics in terms of operators.

Gosson and Hiley warned their reader against overinterpreting their claims:

> We are *not* claiming that we are deriving quantum mechanics from classical mechanics; what we are doing is the following: knowing that quantum mechanics exists, we show that the mathematical formulation of quantum mechanics in its Schrödinger formulation lies within Hamiltonian mechanics. This does not imply that quantum mechanics—as a *physical* theory—can be reduced to classical mechanics.

Similary, Sternheimer has warned against confusing mathematical with physical results:

> A word of caution may be needed here. It is possible to intellectually imagine new physical theories by deforming existing ones ... Nevertheless such intellectual constructs, even if they are beautiful mathematical theories, need to be somehow confronted with physical reality in order to be taken seriously in physics. So some physical intuition is still needed when using deformation theory in physics.

Surely, proofs that classical mechanics, qua mathematical theory, implicitly contains the mathematical apparatus of quantum mechanics do not imply that this apparatus can be interpreted in a physically meaningful manner. Only a Diracian belief that every beautiful mathematics should someday find an application in the world could prompt us to think so. It remains true, however, that quantum mechanics has a striking kind of mathematical necessity, or perhaps even a transcendental necessity if one shares Poincaré's belief that the Lie group structure implied in the mathematical definition of the deformed dynamics is a necessary form of understanding.[33]

3. Quantum logic

An older way of showing the necessity of quantum mechanics is to seek operational reasons for the strange algebra of quantum mechanical observables. The basic idea, introduced by John von Neumann in 1932, is to examine the consequences of the possible incompatibility of measurements performed on a physical system. As every measurement can be regarded as the answer to a series of (compatible) Yes-No questions, the problem boils down to characterizing formal extensions of ordinary (Boolean) logic to propositions whose coordination and disjunction is not always defined. We will first see how Neumann and Garrett Birkhoff developed this idea in a powerful study of 1936. Essentially, they proved that simple natural assumptions for the lattice of propositions made it isomorphic with the lattice of subspaces of a generalized Hilbert space (in the

[33] Ibid., p. 1418. Sternheimer 1998, p. 2. On Poincaré's belief, see chap. x, p. xxx.

case of finite dimension). We will then see how later contributors to quantum logic, especially Constantin Piron in Geneva and George Mackey at Harvard, improved on this pioneering study by consolidating its operational foundation, by extended it to infinite dimension, and by deriving the quantum-mechanical description of states and evolutions.

From Neumann's spectral theorem to a new logic

John von Neumann invented the "logic of quantum mechanics" in the early 1830s, while working on an appropriate mathematical foundation for quantum mechanics. The two pillars of this foundation were the complex Hilbert space of infinite dimension and the spectral theorem for self-adjoint operators in this space. For finite dimension (*N*), it had long been known that every Hermitian matrix *H* admits a sequence of mutually orthogonal eigenvectors $\psi_1, \psi_2, ..., \psi_N$ and a sequence of eigenvalues $\lambda_1, \lambda_2, ..., \lambda_N$ such that $H\psi_n = \lambda_n \psi_n$. In the general (possibly degenerate) case, several eigenvectors may correspond to the same eigenvalue and a given eigenvalue λ_i defines a linear subspace E_i that can have any dimension from 1 to *N*. This is why Hilbert preferred a more intrinsic version of the spectral theorem according to which Hermitian operators with a number *I* of distinct eigenvalues admit the spectral decomposition

$$H = \sum_{i=1}^{I} \lambda_i P_i \, ,$$

where P_i is the orthogonal projector on the subspace E_i. These projectors satisfy $P_i^2 = P_i$; they are Hermitian operators with the eigenvalues 0 and 1; and the associated subspaces are mutually orthogonal. The theorem is easily generalized to bounded operators in an infinite-dimensional Hilbert space.[34]

In quantum mechanics, the operators representing physical quantities are not bounded and they are not even defined in the whole Hilbert space. For instance the momentum operator, $-\mathrm{i}\hbar\nabla$, when applied to a normalized wave function, may lead to a wave function of infinite norm, and its eigenfunctions are plane waves that do not belong to the Hilbert space. Moreover, the set of eigenvalues is not always discrete; it may be continuous or mixed. This is why Neumann generalized the notion of Hermitian operator into that of self-adjoint ("hypermaximal") operator, for which the domain of definition is only required to be dense in the Hilbert space and for which the usual relation of conjugation $\langle \varphi | H | \psi \rangle = \langle \psi | H | \varphi \rangle^*$ holds. He then showed that Hilbert's form of the spectral theorem lent itself to a generalization in which such operators admit the spectral decomposition

$$H = \int_{-\infty}^{+\infty} \lambda P_\lambda \mathrm{d}\lambda \, ,$$

the "spectral measure" $P_\lambda \mathrm{d}\lambda$ being defined so that for any measurable subset Λ of **R**, the integral $P_\Lambda = \int_{\lambda \in \Lambda} P_\lambda \mathrm{d}\lambda$ is an orthogonal projector. This measure is such that the subspaces

[34] See Neumann 1930, pp. 58-59; 1932, pp. 56-58.

associated with the projectors P_Λ and $P_{\Lambda'}$ are orthogonal whenever the subsets Λ and Λ' are disjoint.[35]

The projectors P_Λ were the basis of Neumann's statistical interpretation of quantum formalism. He regarded them as observables with the expectation values 0 and 1. The value 1 corresponds to states for which the value of the observable *H* belongs to the subset Λ, and the value 0 to states for which the value of *H* belongs to the complementary subset. In logical terms, the projector P_Λ is associated with the bipolar proposition: "The value of the observable *H* belongs to the subset Λ." In his influential *Mathematische Grundlagen der Quantenmechanik* of 1932, Neumann commented:[36]

> The relation between the projectors and the properties of a physical system allows for a sort of logical calculus [*eine Art Logikkalkül*] with these projectors. However, in contrast with the calculus of ordinary logic, this calculus is enlarged by the concept of "simultaneous decidability" [*gleichzeitige Entscheidbarkeit*] that is characteristic of quantum mechanics.

Birkhoff and Neumann's proposition calculus

Neumann and Garret Birkhoff developed this new "sort of logical calculus" or "proposition calculus" in a brilliant memoir of 1936. The basic idea is to associate every proposition *a* about the state of a physical system with the invariant subspace A of an associated projector. The relation "*a* implies *b*" is identified with the inclusion of the associated subspaces, the generalized conjunction "meet of *a* and *b*" with the intersection of the associated subspaces, the general disjunction "join of *a* and *b*" with the linear sum of the associated subspaces, the negation of *a* with the orthogonal complement of the associated subspace, the always false proposition "0" with the empty subspace, and the always true proposition "1" with the entire Hilbert space **H**. In symbols, we have

$a \le b$	$a \wedge b$	$a \vee b$	$\bar{a}$	0	1
$A \subset B$	$A \cap B$	$A + B$	$A^\perp$	$\varnothing$	**H**

The relation $a \le b$ is a relation of partial order; the meet $a \wedge b$ is the highest lower bound of *a* and *b* with respect to this relation; the join $a \vee b$ is the smallest higher bound of *a* and *b*. Thus, the set of propositions is what mathematicians call a lattice. This lattice is "orthocomplemented," namely: it has a minimal element 0 and a maximal element 1; and to every *a* corresponds a complement $\bar{a}$ satisfying $\bar{\bar{a}} = a$, $a \wedge \bar{a} = 0$, $a \vee \bar{a} = 1$; if $a \le b$ then $\bar{b} \le \bar{a}$.[37]

The usual operations of logic share the orthocomplemented lattice structure. In addition, they enjoy the property of distributivity:

[35] Ibid., pp. 58-62.
[36] Ibid., p. 253.
[37] Birkhoff and Neumann 1936, pp. 827-830. On lattice theory, cf. Birkhoff 1940; Grätzer 2011.

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \text{and} \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c),$$

which makes the set of propositions a "Boolean lattice." The new proposition calculus is not distributive, as is easily seen for a bidimensional A and one-dimensional B and C. Distributivity only holds within subsets of mutually compatible propositions whose associated projectors commute. The condition for the compatibility of *a* and *b* is

$$a = (a \wedge b) \vee (a \wedge \bar{b}) \text{ and } b = (a \wedge b) \vee (\bar{a} \wedge b),$$

which corresponds to the condition

$$A = (A \cap B) + (A \cap B^{\perp}) \text{ and } B = (A \cap B) + (A^{\perp} \cap B)$$

for the commutativity of the orthogonal projectors onto the subspaces A and B. The compatibility condition is easily seen to be equivalent to the distributivity of the sublattice engendered by *a*, *b*, $\bar{a}$, and $\bar{b}$.[38]

In a Hilbert space of finite dimension, the new calculus enjoys the weaker modular property that Richard Dedekind introduced in a pioneering, late-nineteenth century study of lattices:

$$\text{If } a \leq c, \text{ then } a \vee (b \wedge c) = (a \vee b) \wedge c.$$

This is so because on the one hand, if $A \subset C$, then A and $B \cap C$ are both included in $(A+B) \cap C$, which implies $A + (B \cap C) \subset (A+B) \cap C$; and because on the other hand by repeated use of the identity

$$\dim(X+Y) = \dim X + \dim Y - \dim X \cap Y$$

we have[39]

$$\dim[A + (B \cap C)] = \dim[(A+B) \cap C] = \dim(A+B) + \dim C - \dim(B+C).$$

Neumann and Birkhoff then raise the two following questions:

1) Is every orthocomplemented modular lattice isomorphic to the lattice of subspaces in a Hilbert space or similar construct?
2) Do the axioms of an orthocomplemented modular lattice have a natural physical interpretation?

If these two questions can be answered positively, then quantum mechanics acquires some sort of necessity as a consequence of the natural calculus of propositions concerning tests performed on the system.

From modular lattices to projective geometry

In their answer to the first question, Neumann and Birkhoff rely on a theorem by Birkhoff, according to which any irreducible complemented modular lattice of finite dimension defines a projective geometry of finite dimension. By definition, *a complemented lattice is a lattice for which there exists a 0 and a 1 and for which every element a admits a complement a' such that* $a \vee a' = 1$ *and* $a \wedge a' = 0$ (this operation need

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[38] Birkhoff and Neumann 1936, pp. 830-831, 833 (compatibility).

[39] Ibid., pp. 831-833. Richard Dedekind introduced the notion of modular lattice in 1897: cf. Corry 1996, pp. 121-129.

not be an orthocomplementation). *A lattice is said to be irreducible if there is a third minimal element on the joint of two distinct minimal elements of the lattice* (the rationale of this definition will be explained in a moment). The dimension of the lattice is said to be finite if there is a maximal value for the number of elements in a chain $0 < a_1 < a_2 < .... < 1$. An abstract projective geometry of finite dimension is defined by a set of elements of increasing but bounded "dimension" called point, lines, planes, etc. and satisfying the four following axioms:

P_1: Two distinct points are contained in one and only one line.
P_2: If A, B, C are points not all on the same line, and D and E are two distinct points such that B, C, D are on a line and C, A, E are on a line, then there is a point F such that A, B, F are on a line and also D, E, F are on a line.
P_3: Every line contains at least three points.
P_4: The set of points on lines through any *k*-dimensional element and a fixed point not on the element is a (*k* + 1)-dimensional element, and every (*k* + 1)-dimensional element can be defined in this way.

A convenient model of these axioms is the set of vector subspaces of a finite-dimensional vector space, a one-dimensional subspace being identified with a point, a bidimensional one with a plane, and so forth.[40]

It is easy to see that a projective geometry defines a complemented modular lattice if two elements *a* and *b* are ordered according to the relation "*a* is on *b*." Then the meet of two elements is their intersection, and their join is the smallest element that contains both (or the set of points contained on lines joining points of these two elements). The modularity of this lattice is proved by reasoning similar to that used for the subspaces of a Hilbert space. The 0 of the lattice is the empty set, and the 1 is the maximal element. If *a* is a given element, by P_4 the 1 can be obtained by successive joining of points to this element: $1 = a \vee e_1 \vee e_2 \vee ....$ These points can be chosen outside *a*, since they would otherwise not contribute to the join. Therefore $e_1 \vee e_2 \vee ...$ defines a complement to *a*.

Reciprocally, any irreducible complemented modular lattice of finite dimension (superior to 2) defines a projective geometry. [Proof removed]

The vector space interpretation of projective geometry

The main result reached so far is that any irreducible complemented modular lattice of finite dimension (higher than 2) is a projective geometry. This theorem was essential to Neumann and Birkhoff because it enabled them to exploit familiar results of projective geometry. As was earlier mentioned, the vector subspaces of a finite-dimensional vector space satisfy the axioms of projective geometry. As was known since the previous century, for dimension higher than three the only possible projective geometries are of

[40] Birkhoff 1935.

this kind. For readers not familiar with projective geometry, the following is an outline of the proof.[41]

Firstly, it is easy to see that Desargues's theorem holds in every projective geometry of dimension higher than three: for two non-coplanar triangles ABC and A'B'C' such that the lines AA', BB', and CC' intersect in a common point, the points $\mathrm{AB}\cap \mathrm{A'B'}$, $\mathrm{AC}\cap \mathrm{A'C'}, \mathrm{BC}\cap \mathrm{B'C'}$ are on the same line. Indeed if we call α the plane BCB'C', β the plane ACA'C', γ the plane ABA'B', π the plane ABC, and π' the plane A'B'C', we have

$$\mathrm{AB}\cap \mathrm{A'B'} = (\gamma\cap\pi)\cap(\gamma\cap\pi') = \gamma\cap(\pi\cap\pi'),$$
$$\mathrm{AC}\cap \mathrm{A'C'} = \beta\cap(\pi\cap\pi'), \quad \mathrm{BC}\cap \mathrm{B'C'} = \alpha\cap(\pi\cap\pi'),$$

so that the three intersection points belong to the line $\pi\cap\pi'$. Secondly, for any plane imbedded in a Desarguesian geometry, it is possible to defined the sum and the product of points on a line (for three given reference points O, I, U) in such a manner that the line acquires the structure of a field **K**. The construction, given in figs. 1 and 2, is inspired from perspective drawings in which parallel lines converge on an ideal horizon. The axioms of projective geometry, together with the Desarguesian axiom, imply that the operations defined by this construction are unique (up to a isomorphism depending on the choice of O, U, I) and that they satisfy the algebraic axioms of a field.

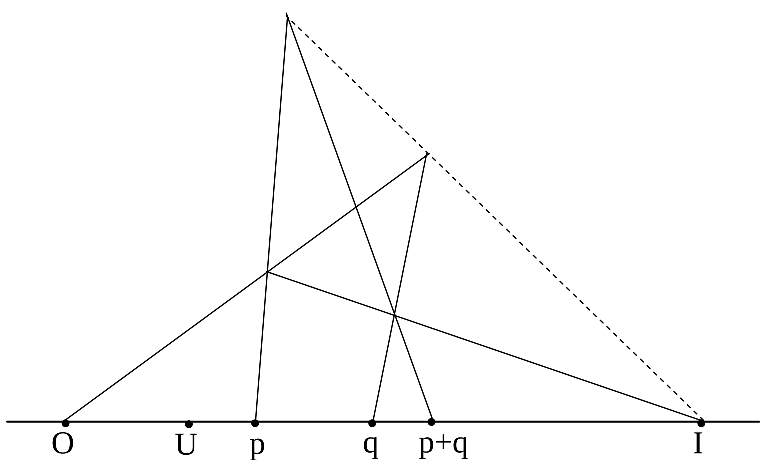

Fig. 1: Construction of the sum p+q of the points p and q on a line with an origin O and an ideal point I. This construction is intuitively justified by regarding the dotted line as the horizon of a perspective drawing.

[41] See, e.g., Garner 1981.

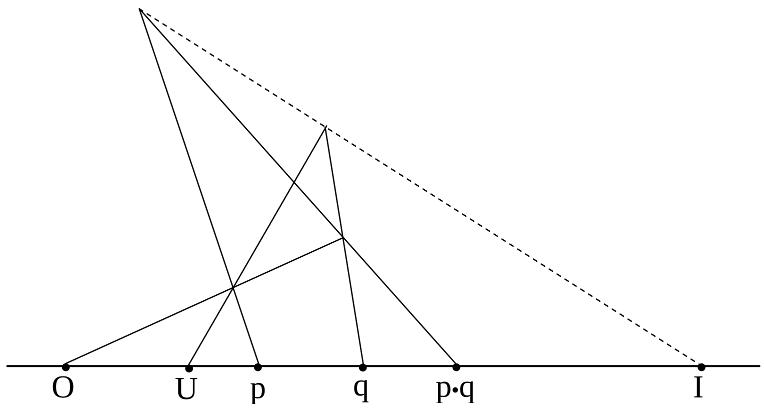

Fig. 2: Construction of the product p•q of two points on a line with an origin O and a unit U.

The construction of fig. 3 then provides the coordinates of a point in a perspective plane. To the point of coordinates x and y we can associate the vector-line of $\mathbf{K}^3$ engendered by the vector $(1, x, y)$. Then a line of the projective plane is easily seen to correspond to a vector-plane of $\mathbf{K}^3$. More elaborate constructions yield similar results for projective geometries of dimension N higher than three. The objects of these geometries are thus interpreted as vector subspaces of $\mathbf{K}^N$.

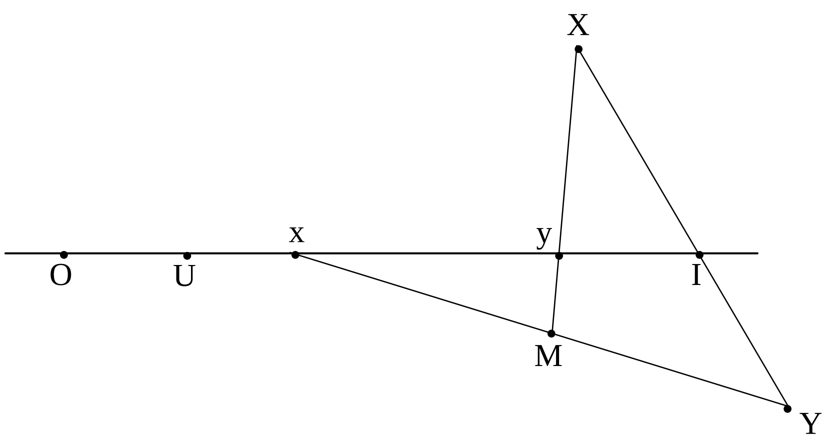

Fig. 3: Construction of the coordinates x and y of a point M with respect to the ideal point I and the ideal axes OX and OY.

From orthocomplemented modular lattices to generalized Hilbert spaces

The field on which the vector space is built is arbitrary. It may even be non-commutative (in which case Pappus's theorem does not hold). We will now see that the orthocomplementation of the lattice associated with the projective geometry brings a restriction on this field. From quantum mechanics, we already know that the vector subspaces of a Hilbert space define a projective geometry whose associated lattice is orthocomplemented. The demonstration of this fact remains unchanged if we replace the Hilbert space by a more general kind of space which I shall call **K*-**space. Such a space is obtained by replacing the field of complex numbers with any field **K** that supports a "star conjugation" with the properties

$$(x+y)^* = x^* + y^*,\ (xy)^* = y^* x^*, \text{ and } x^{**} = x$$

for any two elements *x* and *y* of the field, and if the Hermitian product of two vectors *a* and *b* is replaced with the definite form $<a,b>$ such that

$$<a,b+c> = <a,b> + <a,c>,\ <a+b,c> = <a,b> + <a,c>,$$
$$<a,\lambda b> = <a,b>\lambda \text{ for } <\lambda a,b> = \lambda^* <a,b>,\ <a,b> = <b,a>^*$$

for any three vectors *a*, *b*, *c* and for any element λ of the field. Reciprocally, every orthocomplemented projective geometry of dimension $N \geq 3$ is isomorphic to the set of vectors subspaces of a **K***-space of dimension *N*. Neumann and Birkhoff's proof of this remarkable theorem being somewhat opaque, I will provide a simple proof in the case $N = 3$ [proof removed]. The proof for higher dimension is easily obtained by considering three-dimensional subspaces.

If we combine this result with Birkhoff's earlier result that every irreducible complemented modular lattice defines a projective geometry, we arrive at the conclusion that every irreducible orthocomplemented modular lattice is isomorphic to the lattice of subspaces of a **K***-space. This answers the first question of Neumann and Birkhoff about their new calculus of propositions. The second question concerns the physical interpretation of the axioms of an orthocomplemented modular lattice. Before examining possible replies to this second section, the reader should be warned against a pervasive confusion in the literature on quantum logic.

Is quantum logic a logic?

Birkhoff and Neumann clearly did not intend to replace ordinary logic by their new calculus of propositions. By calling this calculus a "quantum logic," they only meant that it was a mathematically natural extension of the Boolean calculus of ordinary logic. In contrast, their physicist and philosopher readers have often taken the expression "quantum logic" literally. They have placed quantum logic and ordinary logic on the same footing, either because they regarded both kinds of logic as empirically founded, or because they regarded ordinary and quantum logic as both justifiable by a priori means. In analogy with geometry, Hilary Putnam famously advocated the first option in his "Is logic empirical?" of 1968:

> I want to begin by considering a case in which 'necessary' truths (or rather 'truths') turned out to be falsehoods: the case of Euclidean geometry. I then want to raise the question: could some of the 'necessary truths' of logic even turn out to be false *for empirical reasons*? I shall argue that the answer to this question is in the affirmative, and that logic is in a certain sense a natural science.

Putnam took Neumann's quantum logic very seriously and propounded to "*just read the logic off from the Hilbert space.*" In this view, both logic and quantum mechanics would have no a priori necessity, and they would both derive at least in part from experience. Peter Mittelsteadt is a nuanced defender of the second option, which lends quantum logic some a priori necessity. He indeed believes that both the laws of classical and those of quantum logic depend on dialog games or proof trees whose structure depends on pragmatic preconditions of the language of physics:[42]

> [In the operational approach] quantum logic appears as an a-priori structure that is justified more rigorously and under weaker assumptions than the laws of classical logic. This means that first of all the pretended preference of classical is no longer justifiable, and secondly that quantum logic contains less empirical contributions—if at all—than classical logic.

In Kantian terms, both Putnam and Mittelsteadt regard the laws of logic as synthetic judgments, a posteriori synthetic for Putnam, a priori synthetic for Mittelsteadt. They thus contradict a long tradition of regarding these laws as analytic judgments. Philosophers usually do not mind, because they have assimilated Willard Van Orman Quine's attack on the traditional distinction between synthetic and analytic judgments in his famous "Two dogma of empiricism" of 1951. If, pace Quine, logic is purely analytic, one should not confuse it with a calculus of experimental propositions; and one should not try do derive its laws from experience; conversely, one should not try to derive rules of combined experimental tests from the laws of logic. This is the position defended by Jauch:

> The calculus introduced here has an entirely different meaning from the analogous calculus used in formal logic. Our calculus is the formalization of a set of *empirical* relations which are obtained by making measurements on a physical system. It expresses an objectively given property of the physical world. It is thus the formalization of empirical facts, inductively arrived at and subject to the uncertainty of any such fact. The calculus of formal logic, on the other hand, is obtained by making an analysis of the meaning of proposition. It is true under all circumstances and even tautologically so. Thus ordinary logic is used even in quantum mechanics of systems with a propositional calculus vastly different from that of formal logic. The two need have nothing in common.

If, contrary to Jauch's opinion, the truths of logic are synthetic truths similar to those of arithmetic or geometry, a strong quantum-logic project becomes conceivable. This thorny issue may be avoided by pragmatically regarding the logic of ordinary reasoning (including reasoning used to defend a new logic!) as unaffected by the physical calculus of propositions. In this view, the question of the necessity of this calculus is decoupled from the question of its interpretation as a logic; its answer should be sought in the naturalness of the implied physical operations, not in pure laws of thought.[43]

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[42] Putnam 1968, pp. 216, 221; Mittelsteadt 1978; 2004, chaps. 3, 13; 2011, p. 72. Cf. Stachel 1974; Bacciagaluppi 1993, 2009.

[43] Mittelsteadt 1978; Jauch 1968, p. 77.

The physical interpretation of the logical axioms

The question is whether the axioms of an orthocomplemented modular lattice can be embodied in natural physical operations. Let us first consider the case of finite dimension *N*. The lattice properties are easy to justify, because $a \le b$ has a well-defined operational interpretation as the statement that the binary test *b* (Yes-No experiment) yields a positive result if the test *a* has just been performed with a positive result.[44] The axioms of partial ordering are evidently satisfied because every ideal test is assumed to be repeatable and because ordinary implication is transitive. Orthocomplementation and its properties are also easy to justify by the duality of the Yes and No answer to binary tests: $\bar{a}$ is the proposition that the test *a* gives a No answer. Irreducibility holds if and only there are no non-trivial tests that are compatible (non-interfering) with every other test. This property does not hold in quantum systems that obey superselection rules: for instance, the mass of a particle can be measured without interfering with any other measure (in other words, the superposition of states of different mass is not allowed). However, generalization to reducible lattices is unproblematic: as was earlier mentioned, a reducible lattice is the direct product of irreducible sublattices, in conformity with the interpretation of superselection rules in terms of non-combining sectors of the Hilbert space of quantum states.[45]

Modularity is somewhat harder to justify. Neumann and Birkhoff offered a physical argument based on the earlier discussed equivalence of modularity with the existence of a numerical dimension-function such that

- If $a < b$, then $\dim a < \dim b$
- $\dim a + \dim b = \dim(a \vee b) + \dim(a \wedge b)$.

If one forgets about the difference between meet, join, and the usual logical operations, these properties makes $N^{-1} \dim a$ a probability which Neumann and Birkhoff interpret as the a priori statistical weight of the associated quantum state (which is a basic notion of quantum statistical mechanics), or as the probability of a positive outcome of test *a* when nothing is specified as to its preparation.[46] Neumann and Birkhoff nonetheless judge that "it would be desirable to interpret [modularity] by simpler phenomenological properties of quantum physics."[47]

This can be achieved by replacing modularity with two properties introduced by Constantin Piron in his influential dissertation of 1964: weak modularity and atomicity. A lattice is said to be *weakly modular* if and only if for any two elements *a* and *b*, $a \le b$ implies *a* is compatible with *b*. This condition is evidently met in quantum mechanics, because $a \le b$ translates into $P_a = P_a P_b = P_b P_a$ and because compatibility corresponds to $P_a P_b = P_b P_a$. An orthocomplemented weakly modular lattice is called an *orthomodular*

[44] Here and henceforth I use "binary test" or just "test" instead of Jauch's "Yes-No experiment." Although my tests are most commonly called "measurements" in the quantum logic literature, I prefer to reserve "measurement" to a test involving a concrete operation of addition (in conformity with the Helmholtzian definition of chapter x). I loosely use the same letter, say *a*, to represent a proposition and its test.

[45] Cf. Jauch 1968, pp. 124-126.

[46] The latter interpretation is from Jauch 1968, p. 84.

[47] Birkhoff and Neumann 1936, p. 833.

lattice. An *atom* (also called a point because of the geometric interpretation) is a minimal non-zero element of the lattice. An *atomic lattice* is a lattice satisfying the two following axioms:

A_1: Every element contains an atom.
A_2 (covering law): If a and b are elements of the lattice and e an atom, one can never have $a < b < a \vee e$ ($a \vee e$ at most covers a).

Any orthomodular atomic lattice of finite dimension is modular. A proof of this theorem follows.[48] [proof removed]

Let us now see whether weak modularity and atomicity have natural justifications. Operationally, weak modularity corresponds to the condition: if a binary test b has a well-defined result whenever a has just been tested, then the testing sequence a, b, a always yields the same result for the two tests of a (and so do too the two tests of b in the sequence b, a, b). This seems reasonable, because the premise $a \leq b$ intuitively implies that the test b refines our knowledge of the system without destroying knowledge acquired by the test a. For finite dimension, the atomic axiom A_1 holds necessarily since a chain below any given element of the lattice cannot be indefinitely lengthened by inserting a non-zero element under its least element.

The covering law A_2 is less obvious. In his dissertation, Piron remarked that for propositions x compatible with a given proposition a, the correspondence $x \to a \vee x$ fills the sublattice of propositions containing a, for which a plays the role of a zero. This sublattice has a simple physical interpretation: it concerns the tests done on the system when $\bar{a}$ is known to be true of the system, because the test $a \vee x$ is then equivalent to the test x. One should therefore expect the correspondence to turn any atom e of the full lattice (not included in a) into an atom of the sublattice, which means that $a \vee e$ covers a. Piron regarded this remark as a "justification" of the covering law. Alas it cannot be, because weak modularity by itself implies that $a \vee e$ at most covers a if a and e are compatible, and because compatibility is not assumed in the covering law. At best one could try to argue that the validity of the covering law for compatible atoms makes it plausible for incompatible atoms.[49]

It may be noted, however, that even for incompatible a and e, their join is compatible with a, so that $a \vee e = a \vee d$, with $d = (a \vee e) \wedge \bar{a}$. Since d is compatible with a, the covering law will be justified if we can find a physical reason for d being an atom. This is what Piron managed to do a few years later by assuming the existence of repeatable tests for every proposition. An atom e of the lattice being associated with a maximal repeatable test, it can also represent a state of the system produced by this test (a pure state in quantum-mechanical language). Suppose that the system was originally in this state and that a test a then gives a No answer. After this test, we expect the system to still be in a maximally known state, say f, and we expect this state to be determined by the set of tests that do not interfere with it. This set comprises a second test of a (since

[48] Piron 1964, pp. 446 (weak modularity), 448 (atomicity), 460 (other proof of the theorem). Also, the theorem indirectly results from Piron's proof that any irreducible complete orthomodular lattice is isomorphic to a projective geometry, since in the finite-dimensional case the lattice associated to a projective geometry is modular.
[49] Piron 1964, p. 448.

such tests are repeatable), and the test of every proposition x containing e and compatible with a. Indeed the latter kind of test may be indifferently performed before or after the (first) test of a (owing to compatibility interpreted as non-interference)[50] and it obviously does not alter the state e in the former case. Compatibility implies $x = (x \vee a) \wedge (x \vee \bar{a}) \geq (e \vee a) \wedge (e \vee \bar{a}) = b$. Therefore, the state f should be determined by $f \leq b$ and $f \leq \bar{a}$, or by $f \leq \bar{a} \wedge b = (a \vee e) \wedge \bar{a} = d$. This can only be true if d is an atom and $f = d$. Piron thus determined the final state and justified the covering law:

It is important to remark that without this axiom we cannot determine the final state of the system; and although the measurement may be ideal, [without this axiom] the perturbation results in a loss of information, even if we take the response of the system into account.

This argument implies conditional tests (the result of a previous test) and the notion of maximally known (pure) states. More recently and within the context of his and Mittelsteadt's schematization of the empirical proof of propositions, Ernst Walther Stachow has shown that the covering law can be derived from existence of conditional probabilities for tests performed on individual systems. This and others corroborations of Piron's intuition strongly plead of a certain necessity of the covering law, although not a kind of necessity that jumps to the eyes.[51]

Infinite dimension

In sum, we see that in the case of finite dimension, the axioms of an orthomodular atomic lattice correspond to natural expectations about tests performed on a physical system. Unfortunately, usual quantum mechanics requires a propositional lattice of infinite dimension. The easiest way to deal with this difficulty is to assume that quantum mechanics in finite-dimensional Hilbert space is more basic than its infinite-dimensional counterpart and to derive the latter from the former by simply requiring that finite-dimensional subspaces of the latter should have the structure of the former. This procedure is physically justified inasmuch as quantum processes can concretely be restricted to transitions between a finite number of quantum states, as happens for instance in the two-level approximation of atoms interacting with properly tuned radiation. Operationally, a finite dimension of the propositional lattice corresponds to a maximal length for a chain of refinements of any binary test. This can only happen if the propositions correspond to sets of measurements of quantities that can only take a finite number of (sharply defined) discrete values.

For those who do not wish to assume so much from the start, it is necessary to examine the general case of lattices of infinite dimension. The mathematical component of this generalization is not too problematic. Piron accomplished most of it by adding the axiom of completeness according to which the minimal upper bound $a \vee b$ of two elements and their maximal lower bound $a \wedge b$ still exist despite the infinite dimension, and by replacing modularity with weak modularity and atomicity. The latter change is

[50] The test is "ideal" in Piron's sense, namely, it does not modify the outcome of a compatible test.

[51] Jauch and Piron 1969, pp. 847-848; Piron 1976, p. 69 (citation); Stachow 1985. On other justifications of the covering law, see Wilce 2009, §5.

necessary because the lattice of subspaces of a Hilbert space of infinite dimension is not modular[52] and because perfectly conceivable quantum measurements, for instance position measurements, violate modularity.[53] Piron managed to prove that any irreducible complete orthomodular atomic lattice could be represented by the lattice of subspaces of a **K***-space.[54]

The operational embodiment of Piron's mathematical extension is more troublesome. Even the basic lattice operations, the meet and the join, are hard to justify because their empirical realization requires infinitely many operations. For instance, $a \wedge b$ is true if and only if an infinite sequence of alternate tests of *a* and *b* all yield positive results.[55] In 1969 Jauch and Piron tried to circumvent this difficulty by defining propositions as classes of equivalence of experimental Yes-No questions and the product of a family of questions as an arbitrary question of the family.[56] The success of this procedure is doubtful, for it involves potentially infinite classes of questions and an unclear notion of arbitrariness. According to Hans Primas, the best one can do is to regard the lattice structure as mathematically convenient. Then, orthocomplementation and weak modularity can be justified in the same manner as in the case of finite dimension. In contrast, atomicity becomes artificial. Why should there be a limit to the refining of a combination of tests? As Primas notes, atomicity does not hold in powerful axiomatic formulations of classical and quantum statistical mechanics. In the lattice of propositions of quantum mechanics, atoms correspond to pure states. This suggests that atomicity only applies to individual systems (the classical case is controversial). A possible justification for the existence of an atom below an element of the lattice would be the finite dimension of this element, despite the infinite dimension of the lattice. This is not so far, however, from assuming the concrete possibility of finite-dimensional lattices of propositions. Then we may as well begin with such lattices and delay infinite-dimensional generalization until the **K***-space representation has been derived.[57]

States, probabilities, and dynamics

Once a calculus of propositions or quantum logic has been defined, it is tempting to define the state of a system through the list of probabilities for the outcomes of all binary tests on the system. The Harvard mathematician George Mackey did so in 1957 in an attempt to base quantum mechanics on a series of plausible axioms. The central notion of his theory was the probability $p(\mathrm{A}, \mathrm{S}, a)$ for the observable A to take the value *a* when the system in the state S. Stated informally, his first four axioms were:

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[52] Neumann and Birkhoff originally hoped to circumvent this difficulty by restricting the permitted projectors and subspaces to a modular subclass.
[53] See Jauch 1966, pp. 219-221.
[54] Piron 1964. A student of Mackey's, Malcom Donald MacLaren, and two Japanese mathematicians, Ichiro Amemiya and Huzihiro Araki, perfected the proof. Cf. Primas 1981, p. 212.
[55] This has to do with the projector formula $P_{a\wedge b} = \lim_{n\to\infty}(P_a P_b)^n$: cf. Mittelsteadt 1978, p. 20.
[56] Jauch and Piron 1969; Piron 1976, pp. 20-23.
[57] For a lucid discussion of all these difficulties, cf. Primas 1981, pp. 214-219,

1) p is a probability measure in Kolmogorov's sense.
2) The state S is defined by the function $(\mathrm{A}, a) \rightarrow p(\mathrm{A}, \mathrm{S}, a)$, and an observable A by the function $(\mathrm{S}, a) \rightarrow p(\mathrm{A}, \mathrm{S}, a)$.
3) For every observable *A*, one can define an observable $f(A)$ taking the values $f(a)$.
4) Every convex mixture of states is a state.

In the spirit of Neumann's *Grundlagen*, Mackey next introduced "questions," that is, observables that take the values zero and one only. In two additional axioms, he required the existence of the sum of mutually exclusive questions and he associated a question to every bivalued probability measure on the space of states. This allowed him to redefine the theory through the lattice of questions (Neumann's propositions) and through a probability measure on this lattice. In his penultimate axiom, he brutally assumed the lattice of questions to be isomorphic with the lattice of closed subsets of a Hilbert space. In the ultimate one, he required the probability of a positive answer to question to be given by $\mathrm{Tr}\rho P$, wherein *P* is the orthogonal projector on the subset associated with the question and ρ is a positive operator of trace one (a density matrix).[58]

Mackey believed in some "physical plausibility" of all his axioms except the Hilbert-space one. As he did not have in hand the kind of operational justification later given by Piron, he contented himself with showing that this axiom resulted from a most simple and elegant extension of classical logic in which the lattice of propositions would be modular and orthocomplemented in the finite-dimensional case. Mackey of course knew from Neumann and Birkhoff that the latter properties implied the Hilbert-space representation, and he and Shizuo Katukani had made a step toward infinite-dimensional generalization in the modular case. As for the last axiom, he soon heard from his Harvard colleague Andrew Gleason that it was not needed: any probability measure on the subspaces of a Hilbert space could be represented by a matrix density.[59]

For adepts of quantum logic, this is an essential result for it makes the quantum mechanical representation of states a mere consequence of their definition through probabilities of propositions subjected to the rules of quantum logic. Unfortunately, Gleason's proof of his theorem is difficult, and the many attempts at simplifying it have been moderately successful. Fortunately, the theorem may be replaced by the much simpler variant that the British mathematician Paul Busch stated and demonstrated in 2003. Let us begin with a more precise statement of Gleason's theorem in the context of Piron's quantum logic.[60] [proofs omitted]

States being defined by a density matrix, it is natural to represent the evolution of a system by a one to one correspondence between matrix densities ρ and ρ' representing the states of the system at two different times *t* and *t*'. In his Harvard lectures, Mackey developed this idea by further requiring this correspondence to depend on the time difference $\tau = t' - t$ only (uniformity of time) and to preserve convex mixtures (conservation of probability). In symbols, $(\alpha\rho + \beta\sigma)' = \alpha\rho' + \beta\sigma'$ for any two density matrices ρ and σ and for any two weights α and β such that $\alpha \geq 0$, $\beta \geq 0$ and

[58] Mackey 1957.
[59] Mackey 1963, pp. 72-73 (Hilbert-space axiom), 74 (Gleason); 1957, pp. 50-51 (Gleason); Gleason 1957.
[60] Busch 2003. On proofs of Gleason's theorem, cf. Dvurečenskij 1993, pp. 130-131.

$\alpha + \beta = 1$. Obviously, the transformation $\rho' = U\rho U^{-1}$, wherein *U* is a unitary of anti-unitary operator, meets this condition. With the help of a powerful theorem by Richard Kadison on automorphisms in C*-algebras, Mackey proved that reciprocally any mixture-preserving one-to-one mapping of the set of unitary operators onto itself could be generated by a unitary or anti-unitary operator (this operator being defined up to a phase factor). The uniformity of time implies that $U(\tau)$ and $U^2(\tau/2)$ only differ by a phase factor. Since the square of an anti-linear operator is linear, the anti-unitary option is excluded for the evolution operator *U.* [61]

Kadison's proof of his theorem is for experts on C*-algebras. Walter Hunziker has given the following direct proof of Mackey's result. [Proof omitted]

Quantum logical necessity

Quantum logic, seen as a calculus of elementary empirical propositions, rests on very broad and fairly natural assumptions about how we may experiment on a physical system. Its only odd feature is the omission of a most natural assumption of classical measurement: the possibility of eliminating the mutual interference between successive measurements. In other words, quantum logic is an impoverished version of the natural axiomatics of classical measurement. Although some axioms, for instance the existence of a least upper bound or the atomicity of the lattice, have frequently been criticized as too restrictive or too artificial, their necessity is evident in the case of finite dimension of the lattice of propositions. Conceptual and mathematical difficulties are mostly confined to the limit of infinite dimension, which may be postponed until the basic structure of quantum mechanics has been obtained for finite dimension.[62]

At any rate, Neumann and Birkhoff's axioms in the finite-dimensional case and Piron's axioms in the infinite-dimensional case have far-reaching consequences: the lattice of proposition must be isomorphic to the lattice of subspaces of a **K***-space, namely, a generalized Hilbert space built on a field **K** equipped with a kind of conjugation. Quantum mechanics corresponds to the case in which **K** is the field of complex numbers. For this choice, it is possible to derive the matrix-density representation of states defined through the statistics of binary tests. Furthermore, the evolution of a system is determined by a unitary evolution operator in the manner of quantum mechanics.

The quantum logic approach to the foundations of quantum mechanics seduces by its being based on a very simple reduction of experiment to the answer to a series of Yes-No questions. It is also impressive by its ability to (partially) deduce the esoteric Hilbert-space apparatus of quantum mechanics from fairly natural notions. Much of the bad publicity that quantum logic got in some quarters resulted from unnecessary confusion between the calculus of empirical propositions and a genuine logic of thought processes.

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[61] Mackey 1963, pp. 81-82; Kadison 1951; Piron and Jauch favor another approach in which the evolution is assumed to be an automorphism of the lattice of propositions. This leads to the same result by Wigner's theorem, which states that all automorphisms of the lattice of subspaces of a Hilbert spaces are generated by unitary or anti-unitary operators. Cf. Beltrametti and Cassinelli 1981, pp. 252-254.

[62] On later developments of quantum logic and on improvements of its operational grounding, cf. Coecke, Moore, and Wilce 2000; Wilce 2009.

The approach is not without defects, however. In its original form, quantum logic is not able to select the field **K** on which the **K***-space is built. In particular, the field of real numbers or the field of quaternions remains possible. To this day there is no consensus on whether quantum logic can be naturally completed to exclude fields other than **C**.[63]

Another defect of quantum logic is the level of mathematics required to prove its main theorems, as should be expected when mathematicians of Neumann's or Mackey's power get involved. The mathematical burden is alleviated by restricting the reasoning to finite lattice dimension and by substituting simpler proofs to the original ones, as I have tried to do in this presentation. Even so, quantum logic requires lattice-theoretical and projective-geometrical notions unfamiliar to most physicists.

4. Discreteness, probabilities, and information

Quantum logic was not the only attempt to base quantum mechanics on natural axioms. We already mentioned Mackey's attempt, in which probabilistic axioms play an important role. Among other old axiomatics, most noticeable is Günther Ludwig's, which took off in the mid-1950s and reached its mature form in the mid-1980s. Although Ludwig's declared aim was to base quantum mechanics on "physically interpretable axioms," the demands of mathematical rigor and completeness led him into overabundant formalism. In its final form, his theory has not less than seventy-six axioms, most of which are there only for mathematical reasons. Ludwig starts with formal characterizations of preparation and registration procedures, and defines states (ensembles) and observables (effects) through the statistics of these procedures. After imbedding ensembles and observables in Banach spaces, he ends up deriving a lattice of propositions and using Piron's representation theorem in order to reach the Hilbert-space structure of quantum mechanics.[64]

While Mackey and Ludwig shifted the foundational basis from quantum logic to the structure of a probabilistic state space, they failed to improve on the deductive, rational economy of quantum logic. Indeed they both used quantum logic as an important (though not primitive) bridge between their axioms and quantum mechanics. The real turning point in natural quantum axiomatics was a memoir published in 2001 by the British theoretical physicist Lucien Hardy. As we will now see, Hardy changed the axiomatic game by short-circuiting the representation theorems of quantum logic and by instead deriving quantum mechanics "from five reasonable axioms" about probabilistic state space. In his theory, statistical correlation between discrete measurements is the most basic notion. The states of a system are defined through measurement probability distributions, which may be seen as the expression of information content.

Although the mathematics used by Hardy and his followers tend to be simpler than those of quantum logic, they still involve notions unfamiliar to most physicists. For this reason, I begin with an intuitive justification of some the main ideas on the simplest known quantum system: a particle with spin one-half (other degrees of freedom being abstracted away). Then I offer a mathematically light treatment of the general case (with

[63] See the discussion in Mittelsteadt 2011, p. 64.
[64] Ludwig 1983; 1985, pp. 1 (citation), 241 (axioms).

a finite number of discrete outcomes for every measurement), drawing on considerations by Hardy and Claude Comte. The following subsections are devoted to Hardy's theory per se, and to two significant improvements by Borivoje Dakić and Časlav Brukner and by Lluís Masanes and Martin Müller. The last section deals with the axiomatics of Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti, which differs from the former ones by being based exclusively on information-theoretic notions.

The one-half spin system

There is a continuous infinity of possible measurements of this system, giving the angular momentum in any direction of space. In contrast, there are only two possible outcomes for each of these measurements: $+\hbar/2$ and $-\hbar/2$. If the system is found to have the momentum $+\hbar/2$ in a given direction, a subsequent measurement performed in a direction making an angle θ with the former direction will give either $+\hbar/2$ and $-\hbar/2$. Let us repeat the same preparation and the same measurement a great number of times. If p_+ and p_- denote the frequencies of the two possible outcomes, we must have

$$p_+ - p_- = \cos\theta$$

in order that the average angular momentum in the direction θ be equal to the projection of the initial angular momentum on this direction. Indeed by a correspondence argument, we expect the total angular momentum (or magnetic moment) of a large number of spin-particles to behave as the angular momentum of a macroscopic object. Since $p_+ + p_- = 1$, we have[65]

$$p_+ = \cos^2(\theta/2),\ \ p_- = \sin^2(\theta/2).$$

To sum up, the double-valuedness of spin, the spatial character of space measurement, and a correspondence argument together imply the well-known quantum-mechanical expression for the correlations between spin measurements in two different directions. In a different notation, the correlation probability for $+\hbar/2$ spin components in the directions (unit vectors) $\mathbf{u}$ and $\mathbf{u}'$ is

$$p(\mathbf{u},\mathbf{u}') = \frac{1+\mathbf{u}\cdot\mathbf{u}'}{2}.$$

In polar coordinates for which $\mathbf{u} = (\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi)$, we have

$$p(\mathbf{u},\mathbf{u}') = \frac{1}{2}[1+\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')] = \left|\cos(\theta/2)\cos(\theta'/2) + \sin(\theta/2)\sin(\theta'/2)\mathrm{e}^{\mathrm{i}(\varphi-\varphi')}\right|.$$

Introducing a bidimensional Hilbert space with two orthogonal state vectors $|+\rangle$ and $|-\rangle$ corresponding to the spins $+\hbar/2$ and $-\hbar/2$ in the polar direction, the vectors

$$|+_{\mathbf{u}}\rangle = \cos(\theta/2)|+\rangle + \mathrm{e}^{\mathrm{i}\varphi}\sin(\theta/2) \ \text{ and } \ |+_{\mathbf{u}'}\rangle = \cos(\theta'/2)|+\rangle + \mathrm{e}^{\mathrm{i}\varphi'}\sin(\theta'/2)$$

are such that

$$p(\mathbf{u},\mathbf{u}') = \left|\langle +_{\mathbf{u}}|+_{\mathbf{u}'}\rangle\right|^2.$$

[65] A similar reasoning is found in Comte 1996, although Comte uses "homogeneity" (see below p. xxx) instead of correspondence.

We thus see that the full quantum-kinematics of a two-level system derives from a very simple combination of discreteness, symmetry, and correspondence.

Now consider a system of two particles with spin one-half, prepared so that its total angular momentum vanishes (which may be verified by measuring the sum of the momenta of the two particles in any given direction). Suppose that the spin of the first particle is found to be $+\hbar/2$ in a given direction. Then the spin of the second particle must be $-\hbar/2$ in the same direction by conservation. By a similar correspondence argument, the probabilities $P_{\varepsilon\varepsilon'}(\theta)$ of finding $\varepsilon\hbar/2$ and $\varepsilon'\hbar/2$ (with $\varepsilon, \varepsilon' = -1, 1$) for the spins of the two particles in two directions making the angle θ must verify

$$P_{++} - P_{+-} = -\frac{1}{2}\cos\theta\,.$$

Taking into account the normalization

$$P_{++} + P_{+-} = P_{-+} + P_{--} = \frac{1}{2},$$

we get

$$P_{++}(\theta) = \frac{1}{2}\sin^2(\theta/2) \quad \text{and} \quad P_{+-}(\theta) = \frac{1}{2}\cos^2(\theta/2)\,.$$

Again, this is the result given by quantum mechanics, with

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle|-\rangle - |-\rangle|+\rangle\right)$$

for the prepared state, and

$$\langle +|\langle +_\theta| = \langle +| \otimes \left(\cos(\theta/2)\langle +| + \sin(\theta/2)\langle -|\right)$$

for the projecting measurement state in the ++ case. As is well known, these correlations violate the Bell inequalities. Thus, a most astounding consequence of quantum mechanics, the violation of EPR locality, can be derived from a simple combination of discreteness, conservation, and correspondence.[66]

These two arguments cannot really pass for a rational derivation of quantum mechanical laws, for they involve two empirical facts: the existence of a two-level system for which possible measurements are mapped by unit vectors in geometrical space, and the existence of combined spin states for which the total angular momentum vanishes. They nonetheless seem to be pointing to some sort of necessity of the quantum mechanics of two-level systems.

N-level systems

Let us now consider an arbitrary *N*-level system, for which any given maximal measurement performed on the system yields *N* distinct discrete outcomes. We assume that every such measurement is stable by repetition, so that it can be used as a preparation of the system. A system prepared in this manner is said to be in a pure state. Let the system be continuously transformed by some interaction. The outcome of a second measurement performed on this system is necessarily stochastic, because the system cannot jump from one discrete value to another during a continuous transformation. By a

[66] I remember hearing this argument or a similar one from Claude Comte.

correspondence argument, *we expect the repetition of the measurement on a great number of identically prepared systems to yield a well-defined probability for each possible outcome of the measurement* (axiom A_1). For transformations by internal interaction or by application of an external classical field we should expect the system to be in a pure state before a second interaction, because there is no information loss in this process. For transformations by interaction with another system whose final state is undetermined, there is a loss of information so that the first system can only be in a mixture of pure states.

The most general mixture involves every discrete outcome A_n of every possible measurement A, with the statistical weight $\alpha_n(\mathrm{A})$. The corresponding state S is empirically characterized by the probabilities for the various outcomes B_m of every possible measurement B:

$$P(\mathrm{S}, B_m) = \sum_{\mathrm{A},n} \alpha_n(\mathrm{A}) P(A_n, B_m),$$

in which $P(A_n, B_m)$ denotes the correlation between the outcome B_m of the measurement B and the outcome A_n of the measurement A. The sum over A may be discrete or continuous. Unless the functions $P(A_n, B_m)$ are chosen in a special way, the number of such combinations is infinite and an infinite number of choices of the measurement B is necessary in order to determine the state of the system. This is excluded by the correspondence condition that a macro-system made of a large number of copies of the system should have a finite number of effective degrees of freedoms: the value of any associated macro-quantity should be a function of the value of a finite number of macro-quantities. Take the example of spin one-half. For a macro-system made of identically prepared spins, the average spin in the direction **u** should be a function of the average spins along three orthogonal axes:

$$\langle \mathbf{S}\cdot\mathbf{u} \rangle = u_x \langle S_x \rangle + u_y \langle S_y \rangle + u_z \langle S_z \rangle .$$

By correspondence, these ensemble averages should be identical with averages calculated from the probabilities that an individual system be found with the spin $\hbar/2$ in the directions $\mathbf{u}$, Ox, Oy, and Oz respectively. Therefore, only three probability measurements are necessary to determine a spin state.

In conformity with this correspondence argument, we will assume that *a finite number of probability measurements is always sufficient to determine the state of the system, despite the infinity of possible measurements* (axiom A_2). In the sequel, the minimal number of required measurements is called[67] K. This means that a state can be represented by the sequence of probabilities $p = (p_1, p_2, ..., p_r, ..., p_K)$ of a fixed set of K *determinations*, a determination being a measurement performed with a given result (for instance, in the spin case the determinations could be $+\hbar/2$ for S_x, $+\hbar/2$ for S_y, $+\hbar/2$ for S_z). The state p is mixed if and only if there exists two constants λ and μ and two states p' and p'' such that $0<\lambda<1$, $0<\mu<1$, and $p = \lambda p' + \mu p''$ for any r. Since every mixture is allowed, the space of states is a convex, bounded subset of $\mathbf{R}^K$ (for a given choice of the set of determinations). Since pure states cannot be mixtures, they

[67] This K is Hardy's K minus one.

necessarily belong to the boundary of this set. Consequently, the subset of pure states is a manifold of dimension inferior or equal to $K-1$.[68]

We now introduce the additional requirement that *any two pure states can be connected by a continuous reversible transformation within the subset of pure states* (axiom A_3). This requirement is similar to Bohr's old principle of mechanical transformability: it is based on the intuition that a continuously varying action on the system, such as a varying impressed field, cannot induce quantum jumps; and on the intuition that the measurability of a quantity requires its continuous variability. With this complement, the previous assumption has an important consequence: *a mixture of $K+1$ pure states can always be reduced to a mixture of K pure states (with varying choices of the latter states)*(condition A'_2).This may be proved as follows. By definition, a given state S is a $(K+1)$-mixture if and only if it belongs to the polyhedron defined by $K+1$ pure states. By axiom A_3, there is a continuous transformation bringing one of these states continuously to another. There are two possibilities: either the state S remains within the polyhedron defined by the evolving set of pure states, or there is a stage of the transformation at which the state S crosses one of the faces of the polyhedron. In the former case, the state S is a *K*-mixture at the end of the transformation. In the latter case, it is a *K*-mixture at the crossing stage of the transformation. This ends the proof of the desired result. By induction, every state can be represented as a mixture of *K* pure states.

A stronger assumption would require *every state to be a mixture of the N eigenstates*[69] *of a single, properly chosen measurement* (axiom A''_2). For instance, in the spin case the most general mixture yields

$$p_+(\mathbf{u}') = \sum_{\mathbf{u}} \alpha(\mathbf{u})\, p(\mathbf{u},\mathbf{u}') = \frac{1}{2} + \frac{1}{2}\mathbf{u}' \cdot \sum_{\mathbf{u}} \alpha(\mathbf{u})\mathbf{u}$$

for the probability of $+\hbar/2$ being measured in the direction $\mathbf{u}'$.

Setting $a = \left|\sum_{\mathbf{u}} \alpha(\mathbf{u})\mathbf{u}\right|$, $\mathbf{u}_0 = a^{-1}\sum_{\mathbf{u}} \alpha(\mathbf{u})\mathbf{u}$, and $a_\pm = \dfrac{1 \pm a}{2}$,

we have

$$p_+(\mathbf{u}') = a_+ p_+(\mathbf{u}_0,\mathbf{u}') + a_- p_-(\mathbf{u}_0,\mathbf{u}'),$$

in conformity with the stronger requirement. In the 1990s, Claude Comte and Daniel Fivel introduced this requirement as the defining characteristic of quantum-mechanical states. Comte named it the "principle of homogeneity of statistical ensembles" and used it to derive the form of the quantum-mechanical density matrices in the case of spins of any value. Fivel made it the nerve of an impressive derivation of quantum-mechanical transition probabilities on fairly natural operational-probabilistic grounds. We will see that later quantum axiomatics appeal to this principle or to its weakened form A_3, at least

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[68] The dimension of the pure-state manifold is not necessarily $K-1$ because the boundary of the convex state domain may include rectilinear segments within which each state is a mixture of the extremities of the segment. In the case of quantum mechanics, this dimension is $2N-2$ (rays of a *N*-dimensional Hilbert space); and $K = N^2 - 1$ (number of real coefficients of a Hermitian matrix of trace 1 in this Hilbert space); so that the value $K-1$ occurs only in the case $N=2$.

[69] An *eigenstate* of a measurement is a state prepared by this measurement.

implicitly. In the following I adopt the weakened form, which is easier to justify by a priori means.[70]

In the notation introduced at the beginning of this paragraph, a *K*-mixture S is defined by the existence of a sequence S_r of pure states (obtained by selecting a given outcome of a given ideal measurement) determinations such that

$$P(\mathrm{S}, A_n) = \sum_{r=1}^{K} \alpha_r P(\mathrm{S}_r, A_n)$$

for any choice of the measurement A and of the outcome index *n*. In the two-level case ($N = 2$), the two outcomes of any given measurement are complementary and it is sufficient to consider only one of them. We will now investigate the lowest values of the characteristic number *K* in this case. The value $K = 1$ is excluded because it would imply the discreteness of the subset of pure states, in contradiction with axiom A_3. It would also imply that the probability of a single determination is sufficient to determine the state of the system, in conformity with the classical probability theory of coin flipping.

For $K = 2$, the dimension of the pure-state manifold must be 1. By axiom A_3, there is a continuous one-parameter group of transformations acting transitively on this manifold. As is well know, any such group is isomorphic to the additive group of real numbers. Call θ an additive parameter, and label the pure states with this parameter. The probability of finding the system in the state θ' when it has been prepared in the state θ is necessarily of the form

$$P(\theta, \theta') = f(\theta - \theta'),$$

because the transformation $\mathrm{T}_{\theta'}$ can be combined with the determinations M_θ and $\mathrm{M}_{\theta'}$ to yield the determinations $\mathrm{T}_{\theta'}\mathrm{M}_\theta\mathrm{T}_{\theta'}^{-1} = \mathrm{M}_{\theta-\theta'}$ and $\mathrm{T}_{\theta'}\mathrm{M}_{\theta'}\mathrm{T}_{\theta'}^{-1} = \mathrm{M}_0$. The condition A_2' then requires that for any triplet $\theta_1, \theta_2, \theta_3$ of pure states and for any value of the mixing weights $\alpha_1, \alpha_2, \alpha_3$ there exists a pair θ_1', θ_2' of pure states and mixing weights α_1', α_2' such that for any state θ,

$$\alpha_1 f(\theta-\theta_1) + \alpha_2 f(\theta-\theta_2) + \alpha_3 f(\theta-\theta_3) = \alpha_1' f(\theta-\theta_1') + \alpha_2' f(\theta-\theta_2').$$

The general solution of this finite-difference equation is a linear combination of solutions of the form $f(\theta) = \mathrm{e}^{\mathrm{i}k\theta}$. For such solutions, the condition

$$\alpha_1 \mathrm{e}^{\mathrm{i}k\theta_1} + \alpha_2 \mathrm{e}^{\mathrm{i}k\theta_2} + \alpha_3 \mathrm{e}^{\mathrm{i}k\theta_3} = \alpha_1' \mathrm{e}^{\mathrm{i}k\theta_1'} + \alpha_2 \mathrm{e}^{\mathrm{i}k\theta_2'}$$

must hold. Since $\alpha_1 + \alpha_2 + \alpha_3 = \alpha_1' + \alpha_2' = 1$, a possible solution corresponds to $k = 0$ with free choices of θ_1', θ_2', α_1', α_2'. For a non-zero value of *k*, there are three unknowns (for instance θ_1', θ_2', α_1') and two real equations (the real and imaginary parts of the above condition). The problem is possible, and for the same value of the unknowns the choice $-k$ is also possible since the α coefficients are real. For two distinct and non-opposite values of *k*, there are four real equations and three unknowns: the problem becomes impossible (except for accidental choices of the parameters for which the equations are not independent). Therefore the most general solution has the form

$$f(\theta) = a + b\mathrm{e}^{\mathrm{i}k\theta} + c\mathrm{e}^{-\mathrm{i}k\theta}, \text{ or } f(\theta) = a + 2b\cos\theta$$

[70] Comte 1996; Fivel 1994.

because f must be real and because the additive parameter of the transformation group can be redefined so that $k=1$. The constraint $f(0)=1$, $0 \le f(\theta) \le 1$ and the existence of a value of θ for which $f(\theta)=0$ imply $a=1/2$ and $b=1/4$. The end result is

$$f(\theta)=\cos^2\theta/2 .$$

Consequently, the pure states may be represented in a two-dimensional Euclidean vector space as

$$|\theta\rangle = \cos(\theta/2)|+\rangle + \sin(\theta/2)|-\rangle ,$$

with the transition probability

$$P(\theta,\theta') = |\langle\theta'|\theta\rangle|^2 .$$

This is the so-called real-Hilbert-space quantum mechanics.

The next simple choice is $K=3$. In this case, the pure-state manifold is at most bidimensional since pure states belong to the boundary of a convex bounded domain of $\mathbf{R}^3$. The one-dimensional case is excluded, because a convex domain cannot be formed by patching portions of ruled surfaces (whose interior points cannot be extreme) in such a manner that all extreme points belong to the same connected curve (as required by axiom A_3). In the two-dimensional case, the associated transformation group has one-parameter subgroups that leave a given pure state invariant. This group therefore has the same properties as the group of motions of a rigid body around a fixed point in Helmholtz's theory space. By the Helmholtz-Lie argument, this group is isomorphic to the SO(3) group of rotations in $\mathbf{R}^3$. With a proper choice of axes (that is, of basis determinations), the set of pure states becomes a unit sphere. The subset of states belonging to a disk passing through the origin of the sphere can be treated as in the $K=2$ case. Since any two pure states $\mathbf{u}$ and $\mathbf{u}'$ on the unit sphere belong to such a disk, the probability $P(\mathbf{u},\mathbf{u}')$ of finding a system in the state $\mathbf{u}$ when it has been prepared in the state $\mathbf{u}'$ is

$$P(\mathbf{u},\mathbf{u}') = \frac{1+\mathbf{u}\cdot\mathbf{u}'}{2} ,$$

just as in the quantum mechanics of a two-level system. As we have seen in the case of a one-half spin particle, this probability law implicitly contains the Hilbert formalism of quantum mechanics in two (Hilbert-space) dimensions.

In the limited context of two-level systems, the choices $K=1$ and $K \ge 4$ are permitted. In order to exclude these cases, consider systems of every possible (finite) N and their associated freedom $K(N)$. We may physically combine a N-level and a N'-level system and investigate the statistical behavior of the combination. It seems reasonable to assume that *this behavior is entirely determined by the statistics of measurements performed on the two components of the combined system* (axiom A_4). In symbols, this means that the state of the combined system is described by the K' probabilities $P(\mathrm{M}_1'),...,P(\mathrm{M}_K')$ of the determinations $\mathrm{M}_1',...,\mathrm{M}_K'$ of the first component system, the K'' probabilities $P(\mathrm{M}_1''),...,P(\mathrm{M}_K'')$ of the determinations $\mathrm{M}_1'',...,\mathrm{M}_K''$ of the second component system, and of the $K'K''$correlations $P(\mathrm{M}_r',\mathrm{M}_s'')$ with $1 \le r \le K'$ and $1 \le s \le K''$. Consequently, the combined system is described by $K'+K''+K'K''$ measurements. Evidently, the number of levels (distinct measurement outcomes for a given maximal measurement) is the product $N'N''$. Altogether, we have $K(N'N'')+1=[K(N')+1][K(N'')+1]$ for any two integers N' and N''.

As is easily seen, this condition can only be met if

$$K(N)+1=N^{\kappa},$$

wherein κ is an integer. This implies that for $N=2$, all even values of K are excluded. Still another argument is needed to exclude odd values of N higher than three.[71]

For the moment, let us assume[72] that *nature has chosen the smallest value of K compatible with the former axioms* (axiom A_5), namely $K=N^2-1$. Evidently, *N*-level quantum mechanics is compatible with this choice: its most general states are represented by matrix densities, which are positive (therefore Hermitian) operators of trace *one* in a *N*-dimensional Hilbert space. Such matrices have $N-1$ independent real elements on the diagonal, and $N(N-1)/2$ complex conjugate pairs of elements outside the diagonal, which makes a total of $N-1+N(N-1)=N^2-1$ real parameters. We will now examine whether quantum mechanics is the only theory compatible with the former axioms and with the choice $K=N^2-1$.

By axiom A_2, the state S of the system is determined by the sequence of probabilities

$$P(\mathrm{S},\mathrm{M}_r)=p_r$$

for K determinations M_r. For $K=N^2-1$, there exist K unit vectors $|r\rangle$ in a *N*-dimensional Hilbert space and a Hermitian matrix ρ such that

$$\langle r|\rho|r\rangle = p_r \text{ for } 1\le r\le K .$$

Indeed the $|r\rangle$ vectors can be chosen so that real-number linear combinations of the projectors $|r\rangle\langle r|$ span the space of Hermitian matrices, in which case $\langle r|\rho|r\rangle = \mathrm{Tr}\rho|r\rangle\langle r|$ is the *r* covariant coordinate of the operator ρ in the basis of these projectors with respect to the scalar product $(\mathbf{A},\mathbf{B})\to \mathrm{Tr}\mathbf{AB}^{+}$ for any two Hermitian matrices **A** and **B**. For an arbitrary measurement M, the probability $P(\mathrm{S},\mathrm{M})$ must be an affine function of the vector *p* that represents the state S, because the probabilities corresponding to mixtures of two states are the weighted sums of the probabilities corresponding to the individual states. Consequently, there exists a *K*-vector *q* and a constant *C* such that

$$P(\mathrm{S},\mathrm{M}) = q\cdot p + C = \sum_r q_r p_r + C .$$

As the constant C can be absorbed in a redefinition of *q*,[73] this probability can still be expressed by means of the matrix ρ:

$$P(\mathrm{S},\mathrm{M}) = \mathrm{Tr}\rho\mathbf{M}, \text{ with } \mathbf{M}=\sum_{r=1}^{K} q_r|r\rangle\langle r| .$$

In plains words, the determination M_r is represented by the projector $|r\rangle\langle r|$ and an arbitrary determination is represented by a linear combination of such projectors.

This does not prove yet that our theory is equivalent to quantum mechanics, because this equivalence further requires the following:

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[71] As we will see in a moment, this argument belongs to Hardy.

[72] The assumption is Hardy's.

[73] Defining $q_r^{(r)}$ so that $P(\mathrm{S},\mathrm{M}_r) = p_r = \sum_r q_r^{(r)} p_r + C$, we have $P(\mathrm{S},\mathrm{M}) = \sum_r q'_r p_r$ with $q'_r = q_r + 1 - q_r^{(r)}$.

1) the matrix ρ is positive and has trace one,

2) there is a one-to-one correspondence between ρ matrices and states S,

3) pure states or pure measurements are in a one-to-one correspondence with rays in the *N*-dimensional Hilbert space.

In order to establish these properties, we will need to know that *any two-level subspace of the space of states of the system behaves as a two-level system* (axiom A_6). The justification of this condition goes as follows.[74]

Consider a complete measurement A with the outcomes $A_1,...,A_N$. A two-level subspace is defined by the states S for which

$$P(\mathrm{S}, A_n) = 0 \text{ for } 3 \leq n \leq N.$$

Call T a reversible transformation that leaves this subspace invariant. To the determinations M_1 and M_2 corresponding to the outcomes A_1 and A_2 of the measurement A, we may associate the new determinations $\mathrm{TM_1T^{-1}}$ and $\mathrm{TM_2T^{-1}}$. The pairs of determinations obtainable in this manner define a new two-level system whose states belong to the defined subspace (since we generally defined a system by its measurements and since every pair of determinations defines a measurement). Axiom A_1 is trivially satisfied, since the probability $P(\mathrm{S},\mathrm{M})$ for a state S and a determination M belonging to the subsystem is the restriction of the original correlation function to the subspace. Axiom A_2 also holds, since the dimension of a subspace of a finite-dimensional space is necessarily finite. By axiom A_3 applied to the original system, we know there is a continuous sequence of transformations connecting any pair of pure states of the subspace. However, the intermediate pure states do not necessarily belong to the two-level subspace. We will nevertheless assume that axiom A_3 is also satisfied, so that the two-level subspace behaves as a genuine two-level system. This system is empirically realizable if the two-level transformations T can be physically realized. Quantum physicists all know that this is the case: when, for instance, an atom in a given stationary state interacts with monochromatic radiation tuned to the frequency of a possible transition to another state, the associated transformation is very nearly a two-level transformation (Rabi cycles for instance).

We are now equipped to prove the desired properties of the ρ matrix representation of the state of a *N*-level system. We first choose the *N* first determinations M_r so that they correspond to the different outcomes $A_1,...,A_N$ of the same complete measurement A. In addition, we choose the $|r\rangle$ vectors so that the *N* first projectors $|r\rangle\langle r|$ represent the pure states S_r defined by the determinations M_r. This is possible because the formal expression of this requirement is

$$\left|\langle r|s\rangle\right|^2 = P(\mathrm{S}_r, \mathrm{M}_s) \quad \text{for} \quad 1 \leq r \leq N \text{ and } 1 \leq s \leq K;$$

and because the number *NK* of such conditions is always inferior to the number $K(2N-1)$ of real parameters that define K $|r\rangle\langle r|$ projectors in the *N*-dimensional Hilbert space. For any such choice of the *N* first determinations, the trace of the operator ρ is one since the sum of the probabilities of the outcomes $A_1,...,A_N$ must be one.

[74] Again, this axiom belongs to Hardy.

Now consider the subspace of states S such that

$$P(\mathrm{S},\mathrm{M}_r)=0 \text{ for } 3\le r\le N .$$

By axiom A_6, this subspace should behave as a two-level system and its states should therefore be describable by density matrices in a bidimensional Hilbert space. This implies that any projector $|\varphi\rangle\langle\varphi|$ with

$$|\varphi\rangle=\lambda|1\rangle+\mu|2\rangle \text{ and } |\lambda|^2+|\mu|^2=1$$

should represent a possible state S_φ of the system. This state is pure with respect to the subspace. A priori it could still be obtained by mixing states not belonging to the subspace. It is nonetheless pure in the global space because it can be derived by a reversible transformation extending the reversible subspace transformation that generates it from the state $|1\rangle\langle 1|$), and because any state S' related to a pure state S by a reversible transformation T is itself a pure state (since $P(\mathrm{S},\mathrm{M})=1$ implies $P(\mathrm{S}',\mathrm{M}')=1$ with $\mathrm{S}'=\mathrm{TST}^{-1}$ and $\mathrm{M}'=\mathrm{TMT}^{-1}$). We next consider the subspace of states S such that

$$P(\mathrm{S},\mathrm{M}_r)=0 \text{ for } 4\le r\le N \text{ and } P(\mathrm{S},\mathrm{M}_{\overline{\varphi}})=0 ,$$

wherein $\mathrm{M}_{\overline{\varphi}}$ denotes the complementary of the determination associated with S_φ. By the same argument, every projector $|\varphi'\rangle\langle\varphi'|$ with

$$|\varphi'\rangle=\lambda'|\varphi\rangle+\mu'|3\rangle \text{ and } |\lambda'|^2+|\mu'|^2=1$$

should represent a possible pure state of the system. After iterating the argument until we reach the vector $|N\rangle$, we may conclude that every unit vector of the *N*-dimensional Hilbert space defines a pure state of the system.

Consequently, $\langle\psi|\rho|\psi\rangle$ is positive for any (unit) vector $|\psi\rangle$, which means that the operator ρ is positive. Any positive (therefore Hermitian) operator of trace one represents a possible state of the system, since any such operator can be decomposed into a linear combination of the projectors that diagonalize it with positive coefficients adding to one, and since this combination corresponds to a mixture of the corresponding pure states. Lastly, any pure state of the system must be represented by a $|\psi\rangle\langle\psi|$ projector, because every other kind of operator would be mixture of such projectors. This ends the proof that the system is equivalent to an *N*-level quantum system.[75]

To sum up, we have considered systems defined by a set of maximal measurements that can have *N* distinct outcomes.[76] We defined pure states as results of

[75] This proof differs from Hardy's by directly introducing the $N\times N$ matrix ρ in Hilbert space instead of the $K\times K$ probability matrix in determination space.

[76] It is not necessary to assume that all maximal measurements have the same number *N* of outcomes. This follows from the existence of reversible transformations between any two pure states and from the invariance of the state $\hat{\mathrm{S}}$ obtained by uniformly mixing all pure states (see below p. xxx). Indeed, if M_i is the determination associated with the *i*th outcome of a maximal measurement (with $1\le i\le N$), for any fixed determination M_0 there is a reversible transformation T_i such that $\mathrm{M}_i=\mathrm{T}_i\mathrm{M}_0$, so that $P(\hat{\mathrm{S}},\mathrm{M}_i)=P(\hat{\mathrm{S}},\mathrm{T}_i\mathrm{M}_0)=P(\mathrm{T}_i^{-1}\hat{\mathrm{S}},\mathrm{M}_0)=P(\hat{\mathrm{S}},\mathrm{M}_0)$ and $1=\sum_{i=1}^{N}P(\hat{\mathrm{S}},\mathrm{M}_i)=NP(\hat{\mathrm{S}},\mathrm{M}_0)$. The choice of M_0 being

maximal measurements, and arbitrary states as statistical mixtures of such states. We then investigated the probability of the various outcomes of every possible measurement for a given state of the system, and proved that this probability has the form given by *N*-level quantum mechanics if the following axioms hold:

A_1: *The repetition of the same measurement on a great number of identically prepared systems yields a well-defined probability for each possible outcome of the measurement.*

A_2: *A finite number K of probability measurements is always sufficient to determine the state of the system.*

A_3: *Any two pure states can be connected by a continuous reversible transformation within the subset of pure states.*

A_4: *The behavior of a combined system is entirely determined by the statistics of measurements performed on the components of this system.*

A_5: *The true value of K is the lowest value compatible with the former axioms.*

A_6: *Any two-level subspace of the space of states of the system behaves as a two-level system.*

The proof begins with the cases $N = 2$ and $K = 2, 3$, which are treated by means of condition

A'_2 : *A mixture of* $K+1$ *pure states can always be reduced to a mixture of K pure states*, which is a consequence of A_2 and A_3.

This condition could have been replaced by Comte's and Fivel's stronger condition,

A''_2: *Every state is a mixture of the N determinations of a single, properly chosen measurement*,

which however does not directly derive from A_2 and A_3. The proof goes on with the demonstration that axiom A_4 implies that $K = N^{\kappa} - 1$, κ being an integer. The value $\kappa = 1$ being excluded by axiom A_3, nature's choice must be $\kappa = 3$ according to A_5. In this case and for $N = 2$, the system is equivalent to a two-level quantum system. The equivalence for higher values of *N* results from the choice $K = N^2 - 1$ and from axiom A_6

The first three axioms can be justified by correspondence arguments. They warrant that a macro-state of a macro-system made of a large number of copies of the system is described by a finite number (A_2) of well-defined (A_1), continuously modifiable (A_3) macro-quantities. The fourth axiom results from the empiricist requirement that the state of a system should always be accessible by measurement and from the correspondence requirement that measurements, being ultimately expressed in terms of classical quantities, should always be analyzable into measurements performed on the components of the system. This axiom is more obvious in the case of very distant components, for which global instantaneous measurement is not conceivable. The sixth axiom is suggested by the concrete possibility of restricting the transitions of a quantum system to two levels. The fifth axiom is the least satisfying. We will see in a moment that it is not necessary.

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independent of the choice of the maximal measurement, so too must be *N*. Cf. Chiribella, D'Ariano, and Perinotti 2011, p. 16.

Hardy's axioms

In 2001, Lucien Hardy's published his "Quantum theory from five reasonable axioms," which is a rationalist attempt to derive quantum mechanics as a natural sort of discrete probability theory:[77]

> Quantum theory is simply a new type of probability theory. Like classical probability theory it can be applied to a wide range of phenomena. However, the rules of classical probability theory can be determined by pure thought alone without any particular appeal to experiment (though, of course, to develop classical probability theory, we do employ some basic intuitions about the nature of the world). Is the same true of quantum theory? Put another way, could a 19th century theorist have developed quantum theory without access to the empirical data that later became available to his 20th century descendants? In this paper it will be shown that quantum theory follows from five very reasonable axioms which might well have been posited without any particular access to empirical data.

The basic ingredients of Hardy's approach are devices for preparing, transforming, and measuring a system, and states defined by the probabilities of measurement outcomes. He introduces the "dimension" N and the "number of degrees of freedom" K' of the system, which correspond to my N and my $K+1$.[78] His axioms read:

H_1 *Probabilities*. Relative frequencies … tend to the same value (which we call the probability) for any case where a given measurement is performed on an ensemble of n systems prepared by some given preparation in the limit as n becomes infinite.
H_2 *Simplicity*. K is determined by a function of N … and for each given N, K takes the minimum value consistent with the axioms.
H_3 *Subspaces*. A system whose state is constrained to belong to an M dimensional subspace … behaves like a system of dimension M.
H_4 *Composite systems*. A composite system consisting of subsystems A and B satisfies $N = N_A N_B$, $K' = K'_A K'_B$.
H_5 *Continuity*. There exists a continuous reversible transformation on a system between any two pure states of that system.

This list of axioms has much in common with my axioms $A_1,\ldots, A_5$. Axiom H_1 is the same as A_1; H_2 is the same as A_5; H_4 can be derived from A_4, as Hardy himself shows[79]; H_5 is the same as A_3. However, Hardy does not regard the finiteness of K (my A_2) as an axiom; he casually introduces this property by arguing that "Most physical theories have some structure which relates different measured quantities." The subspace axiom H_6, is identical with A_6. Hardy justifies it as follows:[80]

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[77] Hardy 2001, p. 1.
[78] Hardy's preference for K' over K comes from his including a probability for the system not to be detected by the measuring device.
[79] Hardy 2001, p. 14.
[80] Ibid., pp. 10, 14.

> This axiom is motivated by the intuition that any collection of distinguishable states should be on an equal footing with any other collection of the same number distinguishable states. In logical terms, we can think of distinguishable states as corresponding to propositions. We expect a probability theory pertaining to *M* propositions to be independent of whether these propositions are a subset or some larger set or not.

Hardy regards all his axioms as "natural" from the point of view of the theory of probabilities. The axiom A_5 of continuity is the one that excludes classical probability and forces us to adopt quantum probability theory if we accept the simplicity axiom. Hardy's justification of the continuity axiom reads:

> Given the intuition that pure states represent definite states of a system we expect to be able to transform the state of a system from any pure state to any other pure state. It should be possible to do this in a way that does not extract information about the state and so we expect this can be done by a reversible transformation.

Implicitly, this is a correspondence argument because the idea of a definite state that can be transformed continuously is a classical idea. In my opinion, correspondence arguments are the true justification of most of Hardy's axioms. Indeed the natural character of an axiom from a probability-theory point of view or from an information-theory point of view does not imply its natural character from a physical point of view. For instance, definite probabilities of discrete outcomes may be natural for a probability theorist and continuous information-preserving transformation may be natural for an information theorist; and yet their combination leads to the quantum weirdness of superposed and entangled states. The reason is that the simultaneous, concrete realization of these two axioms involves an odd mixture of continuity and discontinuity, in a manner imposed by the correspondence principle.[81]

Hardy's derivation of quantum probability theory from his axioms is more rigorous but more difficult than the naïve derivation given in the previous subsection. It begins with a thorough analysis of the structure of probability relations in the vector space of *K*-determinations, which Hardy call "fiducial measurements" (he does not assume pure measurements and pure states from the start). He exploits this structure to derive the probability relations in the case $N=2$, $K'=4$ (the reasoning involves more advanced group theory than my elementary presentation). He proves the relation $K'=N^{\kappa}$ from axiom H_4. Lastly, he sets $\kappa=2$ and uses the subspace axiom H_3 to construct quantum probability theory at any *N* from the $N=2$ case. He does this mostly in the fiducial vector space and he leaves the correspondence of the resulting $K'\times K'$ probability matrices with quantum-theoretical density matrices to the end. Lastly, Hardy shows that his axioms are compatible with two kinds of transformations for closed systems: reversible, probability-conserving transformations represented by unitary transformations in the quantum-theoretical *N*-dimensional Hilbert space, and irreversible

[81] Ibid., p. 15.

transformations associated with measurements and represented by projectors in Hilbert space.

Dakić and Brukner

The clarity and elegance of Hardy's derivation and the simplicity of his axioms have attracted much legitimate attention. An evident defect of this derivation is the artificial character of the simplicity axiom H_2. Hardy himself wondered about the possibility of more complicated theories involving exponents κ higher in the relation $K = N^{\kappa} - 1$.[82] In 2009, the Viennese theorists Borivoje Dakić and Časlav Brukner answered this question negatively by showing that K could not exceed three in the two-level case $N = 2$. They relied on a different system of axioms:[83]

D_1 (Information capacity): An elementary system has the information carrying capacity of at most one bit. All systems of the same information carrying capacity are equivalent.

D_2 (Locality): The state of a composite system is completely determined by local measurements on its subsystems and their correlations.

D_3 (Reversibility): Between any two pure states there exists a reversible transformation.

The axiom D_2 is the same as axiom A_4 and it is directly related to Hardy's axiom H_4. The axiom D_3 is a much weakened form of axioms A_3 or H_5, for it does not assume the existence of a continuous sequence of transformations gradually bringing the first pure state to coincide with the second. This axiom warrants that a compact group acts transitively on the space of pure states. Its weakness is compensated by the strength of axiom D_1, which in fact contains two subaxioms:
D_1': An elementary system has the information carrying capacity of at most one bit.
D_1'': All systems of the same information carrying capacity are equivalent.
Since the information carrying capacity is nothing but the number N of distinct outcomes of a maximal measurement, axiom D_1'' is an information-theoretic rephrasing of (a generalization of) Hardy's subspace axiom H_3.[84] Dakić and Brukner translate their information-theoretic axiom D_1' into "any state of a two dimensional system can be prepared by mixing at most two basis (i.e. perfectly distinguishable in a measurement) states." Here we recognize the Comte-Fivel principle A_2'' in the case $N = 2$. Apparently unaware of Comte's work, Dakić and Brukner borrowed axiom D_1' from their Viennese colleague Anton Zeilinger.

In a celebration of Danier Greenberger's sixty-fifth birthday, Zeilinger suggested to base quantum theory on the principle that "*An elementary system represents the truth*

[82] Hardy 2001, p. 13.
[83] Dakić and Brukner 2009.
[84] The authors credit Grinbaum 2007 for the rephrasing.

value of one proposition" or, equivalently, that "*An elementary system carries 1 bit of information.*" From this principle he derived randomness and entanglement:

> An elementary system can only give a definite result in one specific measurement. The irreducible randomness in other measurements is then a necessary consequence. For composite systems entanglement results if all possible information is exhausted in specifying joint properties of the constituents.

For instance, there can only be one direction of spin measurement for which the spin of a particle can have a definite value because the spin state can only contain the reply to a single Yes-No question. In any other direction, the result of the measurement must be random. For a composite system of two spins, the joint property that the two particles have the same spin in one direction and the other joint property that the two particles have the same spin in another direction exhaust all possible information since the global system has two bits. The answer to other questions, for instance about the spin of one of the particles in a given direction, should be random: this is the signature of an entangled state.[85]

Seduced by this reasoning, Dakić and Brukner turned Zeilinger's information-theoretic principle into the most potent axiom of their theory. Their first remarkable result is that the axioms D_1' and D_3 are sufficient to determine the probability theory for $N = 2$ and any given value of *K*. Here is a simplified proof.[86] [Proof omitted]

There remains to be proved that $K = 1$ (classical probability theory) and $K = 3$ (quantum mechanics) are the only two choices compatible with the axiom. By Hardy's argument on product spaces, we already know that even values of *K* are excluded. In a highly ingenious manner, Dakić and Brukner prove that their axioms exclude any value of *K* higher than 3. Their argument goes as follows. [proof omitted]

Since the even value $K = 2$ is already excluded, we are left with the options $K = 1$ (classical probabilities) and $K = 3$ (quantum probabilities). By Hardy's consideration of subspaces, we can then show that the states of any *N*-level system can be represented by a quantum-mechanical density matrix.

Dakić and Brukner's proof of the impossibility of $K \geq 4$ is easily adapted to Hardy's axiomatics, for it only relies on the subspace axiom and on the representation of the pure states of a two-level system by a $(K - 1)$-dimensional sphere. The latter representation can be derived from Hardy's axioms, without appeal to axiom D_1', by exploiting the possibility of representing any compact group by orthogonal matrices.[87] Dakić and Brukner nonetheless judge their axiomatics superior, because it does not require the continuity assumption (for the transformation group) and because it is compatible both with classical and with quantum probabilities. The continuity assumption is only necessary to exclude the classical option.

One may still prefer Hardy's axioms, because they can be justified by correspondence arguments whereas the information capacity axiom seems hard to swallow. Why after all should every two-level system be assimilated to a one-bit

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[85] Zeilinger 1999, pp. 635, 631.
[86] Dakić and Brukner 2009, pp. 4-6.
[87] See Hardy 2001, appendix 3.

information facility? Is not it highly unnatural to assume, when there is a continuum of possibilities of measurement, that every state of the system can be obtained as a mixture of the outcomes of a single measurement? In 2010, Lluís Masanes and Martin Müller replaced this axiom with the "requirement" that in two-level systems "all mathematically well-defined measurements are allowed by the theory" or that "all tight effects correspond to allowed measurements." [I omit the precise mathematical expression of this condition] We will now see that Masanes and Müller's requirement can replace the Comte-Fivel-Zeilinger axiom D_1' in the derivation of the fact that every point of the sphere $X^2 = 1$ defines an allowed measurement for a two-level system.[88] [proof omitted] Another possible replacement for the axiom D_1' is Hardy's continuity, that is, the possibility of gradually transforming pure states. [proof omitted]

Chiribella, D'Ariano, and Perinotti

In the wake of Hardy's seminal axiomatics, there has been a growing tendency to formulate and justify the axioms of quantum mechanics by information-theoretical means. This is a natural evolution considering the present importance of researches on quantum-mechanical information processing and quantum computing. In 1990, John Archibald Wheeler famously expressed the "It from bit" program for reducing physics to the processing of information. Although this sort of reductionism has often been criticized, it had inspired a few arguments for the information-theoretic necessity of quantum theory. The first of these is found in a memoir of 2003 by three philosophers of physics, Rob Clifton, Jeffrey Bub, and Hans Halvorson (CBH). The gist of their argument is a proof of the three following facts:

1) The impossibility of supraluminal communication between two systems entails the commutativity of the associated albebras of observables.
2) The impossibility of perfectly broadcasting the information contained in an unknown physical state entails the non-commutativity of the algebra of observables of an individual system.
3) The impossibility of unconditionally secure bit commitment entails the existence of entangled states.

The first impossibility (micro-causality) is a mere consequence of relativity theory; the second and third impossibilities are well-known consequences of quantum mechanics applied to quantum cryptography. CBH express their consequences in the language of C^*-algebras, that is, generalizations of the operator algebra on Hilbert spaces meant to encompass every past and future physical theory (see above p. xxx).[89]

[88] Masanes and Müller 2010, pp. 1, 5.
[89] Wheeler 1990. Clifton, Bub, and Halvorson 2003, pp. 2-3. The no-broadcasting theory is an extension of the no-cloning theorem of William Wootters, Wojciech Zurek, and Dennis Dieks (1982) to mixed states. Bit commitment is a cryptographic notion defined by Gilles Brassard, David Chaum, and Claude Crépeau in 1988. On the early history of quantum information, cf. Kaiser 2011. For a criticism of information-theoretic reductionism, cf. Deutsch 2003.

No matter how interesting this result may be as in information-theoretic characterization of quantum theory, it cannot pass for an argument for the necessity of quantum mechanics. There are three reasons for that. Firstly, the C^*-algebraic framework is much too abstractly mathematical to pass for an a priori natural frame in which to formulate physical theory.[90] For a rationalist exploitation of CBH's result one would first need to derive this framework from simple operational considerations, which does not seem easier as deriving the Hilbert space structure of quantum mechanics. A second shortcoming has to do with the contents of CBH's information theoretic principles. Even if they could be shown to be natural from an information-theoretic point of view, this would not make their physical realization in elementary systems more natural. Thirdly, CBH do not prove that quantum mechanics results from their principles. What they construct is a generalized quantum theory defined by a C^*-algebra satisfying the algebraic constraints that derive from their information-theoretic principles. They want this generality because they have in mind situations (quantum field theory, quantum gravity) in which it may be needed.

Very recently Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti (CDP) have offered an "informational derivation of quantum theory" that does not have the first and third of the defects of CBH's derivation. CDP arrive at quantum mechanics, and they do that in a purely operational framework based on probability distributions for circuits resulting from the connection of physical devices:

> Our principles do not refer to abstract properties of the mathematical structures that we use to represent states, transformations, or measurements, but only to the way in which states, transformations, and measurements combine with each other.

CDP give the following informal statement of their axioms:[91]

C_1 *Causality*: the probability of a measurement outcome at a certain time does not depend on the choice of measurements that will be performed later.
C_2 *Perfect distinguishability*: if a state is not completely mixed (i.e., if it cannot be obtained as a mixture from any other state), then there exists at least one state that can be perfectly distinguished from it.
C_3 *Ideal compression:* every source of information can be encoded in a suitable physical system in a lossless and maximally efficient fashion. Here *lossless* means that the information can be decoded without errors and *maximally efficient* means that every state of the encoding system represents a state in the information source.
C_4 *Local distinguishability*: if two states of a composite system are different, then we can distinguish between them from the statistics of local measurements on the component systems.
C_5 *Pure conditioning:* if a pure state of system AB undergoes an atomic measurement on system A, then each outcome of the measurement induces a pure state on system B.
C_6 *Purification postulate*. Every state has a purification. For fixed purifying system, every two purifications of the same state are connected by a reversible transformation on the purifying system.

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[90] The same criticism is found in Grinbaum 2007, p. 402.
[91] Chiribella, D'Ariano, and Perinotti 2011, pp. 2, 3.

The causality axiom C_1 is so evident that all other axiomatizers assumed it without stating it. The local distinguishability axiom C_4 is a rewording of A_4 or D_2. The other axioms are more original. The purification postulate means that every state of a system may be regarded as the marginal state of a subsystem of a larger system that is in a pure state. CDP note the affinity with Schrödinger's remark of 1935:

> An optimal knowledge of the whole does not imply an optimal knowledge of its parts—that is the whole mystery. I would not call that *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought.

Somewhat artificially, CDP include the existence of reversible transformations between any two pure states in the purification postulate. The axioms C_1, C_2, and C_3 serve to derive the duality between pure states and pure determinations as well as the representation of any state as a convex mixture of pure states; whereas in my simplified approach these facts are trivial consequence from the definition of general states as mixtures of states produced by maximal measuring devices. Axioms C_5 and C_6 sustain a proof that $K = N^2 - 1$ and allow the derivation of the density-matrix representation. Interestingly, CDP do not need the subspace axiom. They deduce the equivalence (up to a reversible transformation) of all systems with the same dimension from their own axioms.[92]

In the case of two-level systems ($N = 2$), the axiom C_2 of perfect distinguishability results from Masanes and Müller's requirement that "all mathematically well-defined measurements are allowed by the theory." [proof omitted] This axiom C_2 may replace the latter requirement in the proof that every point of the unit sphere in K-space defines a pure state. [proof omitted]

CDP's proof that axioms C_5 and C_6 imply the relation $K = N^2 - 1$ relies on a difficult and lengthy exploitation of probabilistic teleportation schemes. A much simpler proof can be given by applying the pure-conditioning axiom C_5 directly to the case of the composite system made of the two-level systems I and II. [proof omitted]

Hardy axiomatics and necessity

Let us recapitulate the various deductions given in this paragraph. We started with the definition of a system and its pure states by repeatable (maximal) measurement operations that have a fixed number N of discrete outcomes. The selection of a single measurement outcome is what I call a determination, and it is associated to a pure state. In general, the state S of a system, be it or not the result of a determination M, gives the probability $P(\mathrm{S}, \mathrm{M}')$ for the determination M'. A first axiom requires these probabilities to be well defined. A second axiom allows a finite number of such probabilities to

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[92] Ibid., pp. 2 (on Schrödinger), 29 (no subspace axiom); Schrödinger 1935, p. 555. The purification postulate directly implies the no-cloning and the impossibility of bit commitment assumed by CBH.

determine the state. The minimal value of this number is called *K*; a possible choice of the *K* defining determinations is called a *K*-determination, and the associated probabilities are denoted $p_r = P(\mathrm{S}, \mathrm{M}_r)$, with $r = 1,..., K$. The probability $P(\mathrm{S}, \mathrm{M}')$ may be regarded as a function of the vector $p = (p_1, p_2,..., p_K)$ characterizing the state S and of the vector p' characterizing the pure state attached to the determination M'. Since states and determinations can be mixed and since probabilities add up by mixing, the probability $P(\mathrm{S}, \mathrm{M}')$ must be a biaffine function of the vectors *p* and *p*'.

By a third axiom, there exist reversible transformations between any two pure states. This implies the existence of a "totally mixed" state $\hat{\mathrm{S}}$ for which the probability $P(\hat{\mathrm{S}}, \mathrm{M}')$ is the same for any choice of M'. In the case $N = 2$, the associated vector is $\hat{p} = (1/2, 1/2,..., 1/2)$, and it is advantageous to introduce the Bloch vector of coordinates $x_r = 2(p_r - \hat{p}_r) = 2p_r - 1$. The *K*-determination may be chosen so that the biaffine probability function takes the form

$$P(\mathrm{S}, \mathrm{M}') = \frac{1}{2}(1 + x \cdot x' + x \cdot \mathrm{A}x'),$$

wherein A denotes a skew-symmetric matrix. If S and M are associated with the same pure state of vector *x*, we have $P(\mathrm{S}, \mathrm{M}) = 1$, hence $x^2 = 1$. In order to prove that reciprocally every point of the unit sphere represents a pure state, we need another axiom. Hardy relies on the continuity axiom for the transformation group, Dakić and Brukner rely on the Comte-Fivel-Zeilinger axiom that two-level systems carry one bit of information, Masanes and Müller on the axiom that every tight effect is a possible measurement, CDP on the axiom of perfect distinguishabilly of states belonging to the boundary of the convex state domain.

If our purpose is to show the necessity of quantum mechanics, we should pick the most natural of these axioms. Of course, every author believes his axiom to be the most natural: Hardy expects the continuity of quantum transformations by analogy with the continuity of classical evolutions; Dakić and Brukner believe nature to be made of quantum bits; Masanes and Müller regard tight effects as innocent mathematical idealizations of concrete measurements ("mathematically well-defined measurements"); CDP pride themselves over phrasing their axioms in purely information-theoretic and operational terms.

Dakić, Brukner, and CDP may be criticized for conflating naturalness with information-theoretic simplicity: Why should physics be reduced to transfers of bits? Is not there a huge gap between the abstract idea of quantum bits and their physical realization? Masanes and Müller similarly confuse mathematical simplicity with physical plausibility: Why should their "tight effects" be plausible idealizations of concrete measurements? The main advantage of their axiom is that it leads to the desired result (the Bloch sphere of pure states) in the most direct manner. We are left with Hardy's continuity, whose necessity derives from a more stringent correspondence argument. Some liberty in the choice of axioms is nonetheless welcome as a further indication of the necessity of the consequences.

Once the domain of pure states is known to be the unit sphere, it is easy to prove the existence of a transformation that permutes the states *x* and *x*'. The invariance of the probability function under this transformation requires the vanishing of the skew-symmetric matrix A and the probability function is simply given by

$$P(\mathrm{S},\mathrm{M}') = \frac{1}{2}(1 + x \cdot x') .$$

In order to arrive at the Bloch-sphere representation of two-level quantum mechanics, there remains to prove that $K = 3$. The axiom that the state of a combined system is characterized by the statistics of the determinations of its components leads to the relation

$$K = N^{\kappa} - 1,$$

which excludes all even values of *K*. A further restriction is reached by considering the combination of two two-level systems. On the one hand, a restriction of Hardy's subspace axiom, according to which certain subspaces of states can be regarded as the states of a two-level systems, implies the existence of entangled states for the combined system. On the other hand, by Dakić and Brukner's argument entangled states are impossible for *K* larger than three. The leftover possibilities are $K = 1$, which corresponds to classical probability theory, and $K = 3$, which corresponds to quantum probability theory. Hardy's continuity then excludes the first option. For an *N*-level system, we must have $K = N^2 - 1$, which is the numbers of degrees of freedom compatible with the representation of states by a matrix density in *N*-dimensional Hilbert space. As we saw, the adequacy of this representation can be proved by multiple application of the subspace axiom. This ends the proof that Hardy's axioms or the variants by Dakić, Brukner, Masanes, and Müller lead to quantum probabilities.

A strikingly simple derivation of $K = 3$ for two-level systems is obtained by replacing the subspace axiom with two of CDP's assumptions, the existence of entangled states (or the stronger postulate of purification) and the pure conditioning axiom. The latter axiom, according to which for a combined system in a pure state a determination of one subsystem implies a pure state of the other subsystem, is fairly natural. It is a particular case of a broader axiom that would require any incomplete measurement of a system originally in a pure state to leave the system in a (generally different) pure state.[93] In other words, if we have maximal information on a system, we still have maximal information on this system after performing an ideal (yet incomplete) measurement. Unfortunately, the purification postulate or the resulting existence of entangled states is harder to swallow.[94] To assume this postulate is to admit in the very basis of the theory the quantum oddities deplored by Schrödinger. As explained by CDP, the true advantage of their approach is its providing direct illuminating links between the now information-theoretic axioms of quantum mechanics and the various quantum-information theorems to which physicists have lately devoted much attention.[95]

CDP's charge that other axiomatics always involve uninterpreted mathematical assumptions seems excessive.[96] It certainly applies to Ludwig's old axiomatics, despite Ludwig's intention to provide physically justified axioms; it also applies to Masanes and Müller's "tight effect" axiom; but it does not truly apply to Hardy's approach because his only uninterpreted axiom, the simplicity axiom, is now known to be unnecessary; and the

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[93] As we earlier saw (pp. xx), this supposition is made to justify the covering law in quantum logic.

[94] CDP (2011, p. 2) try to justify the purification principle by having it express the possibility of reducing thermodynamic irreversibility to reversible interaction with an uncontrolled environment.

[95] Chiribella, D'Ariano, and Perinotti 2011, p. 38.

[96] Ibid., p. 2.

charge has no grip on Dakić and Brukner's axiomatics. Altogether, an improved version of Hardy's axiomatics provides a convincing demonstration that the consistent melding of the discontinuity of measurement results with the continuity of measurement possibilities leads to the density-matrix representation of physical states.

There are three limitations to this class of necessity arguments. The first is the assumption of a finite value for the maximal number N of distinct measurement outcomes. This is not a very serious limitation, because in the laboratory quantum processes only involve finite-dimensional Hilbert subspaces (as a consequence of effective infrared and ultraviolet cutoffs). The second limitation concerns the evolution of systems. So to say, the axiomatics of Hardy and his followers only provides the kinematics of quantum mechanics, that is, the representation of physical states. It does not tell us how the states evolve, except that probability should be conserved. In the continuous case in which no measurement is performed, the latter property implies the existence of a Hamiltonian operator from which the evolution derives. Hardy regards the precise expression of the Hamiltonian as a contingent fact to be drawn from experience. However, since his axiomatics implicitly involves correspondence arguments, it would seem natural to extend kinematic correspondence to dynamic correspondence in order to arrive at the usual quantization rules in Heisenberg's or in Schrödinger's form.

The third and most fundamental limitation to a rationalist exploitation of Hardy's axiomatics is inherent in the assumption of strictly discrete (ideal) measurement outcomes. There is no direct empirical difference between an isolated discrete value and a very narrow continuous spread of values around a central value. Yet the two options seem lead to very different intuitions of the possible correlations between successive measurements: only in the first option does one expect well-defined probabilities for these correlations; in the second option, the fine structure of the spectrum of possible value should naturally affect the correlations. In sum, Hardy's axiomatics shows the necessity of quantum mechanics to the extent that the discontinuity of measurement results is judged necessary. Historically, the latter necessity has sometimes been regarded as empirical, for instance as a consequence of the discreteness and universality of atomic spectra; and sometimes as intertheoretical, as the only escape from the paradoxes regarding the interaction between radiation and a large assembly of atoms (infrared catastrophe). These arguments in favor of discontinuous measurement outcomes are not as compelling as the deduction of quantum mechanics from this discontinuity in axiomatics à la Hardy. In the present state of this approach, we should probably content ourselves with the insight that quantum discontinuity, if it is admitted as a fundamental feature of the microworld and if it is complemented with natural axioms concerning the relation between micro- and macro-world, necessarily leads to quantum mechanics as we know it.

We are now in a position to compare the Hardy kind of axiomatics with quantum logic. A first difference is the manner in which non-classicality is introduced. In quantum logic, the classical reference is the Boolean logic of binary measurement; what causes departure from that logic is the admission of incompatible measurements. In Hardy axiomatics, the classical reference is the classical theory of probabilities of discrete events; what causes departure from this theory is the continuity of measurement possibilities. The classical reference being different, it would not make much sense to say that one approach better justifies departure from classicality than the other. Rather, we should compare the manner in which the two approaches purport to derive quantum mechanics.

Let us first compare the crucial ingredients of these derivations. In the quantum logic approach, the most evident quantum-like ingredient is the assumption of incompatible measurements; in the Hardy approach, it is the discreteness of measurements outcomes combined with the continuity of measurement possibilities. At first glance, quantum logic seems more economic, since incompatible measurements are easier to conceive than an intrinsic discontinuity of physical quantities. This difference is tenuous, however. If we believe in Bohr's intuition of quantum discontinuity, the discreteness of physical quantities and the incompatibility of measurements both result from the existence of the quantum action: a measurement generally implies a finite and uncontrollable perturbation of its object, perturbation that randomly affects the result of a subsequent measurement of a correlated object. Moreover, it is doubtful that quantum logic can truly dispense with an assumption of discreteness. As we saw, it is only in the discrete, finite-dimensional case that its axioms are natural enough.

Another difference in the two kinds of derivations of quantum mechanics is the nature of the employed mathematics. For the average physicist, the mathematics of quantum logic is exotic as it involves deep interconnections between lattice theory, projective geometry, and generalized Hilbert spaces. The mathematics of Hardy axiomatics is simpler on average. When it gets more difficult, it is in its reliance on the theory of Lie group, which is abundantly used by modern theoretical physicists. By restricting itself to purely logical axioms, quantum logic has raised the mathematical stakes much higher than Hardy axiomatics.

A last and most decisive difference is the degree in which the two approaches succeed in deriving quantum mechanics. In this respect, quantum logic is losing because it allows for generalizations of quantum mechanics in which the field of complex numbers is replaced by other fields. The fuller success of Hardy axiomatics seems to result from its reliance of axioms regarding composite systems and subsystems. The weaker achievement of quantum logic seems to result from the lack of any such axiom in its (original) foundation. This inferiority is not a sufficient reason to condemn quantum logic: if may well continue to be productive in the golden triangle of mathematics, physics, and philosophy. Yet, globally judging from the number and naturalness of the axioms, from the accessibility of the mathematics, and from the fullness of the deductions, Hardy axiomatics appears to be a better rational derivation of quantum mechanics.

Conclusions

The history of quantum theory is a first remedy for the mathematical abruptness of standard quantum mechanics. It provides an understandable genesis of both the matrix and the wave form of this theory. However, the historical development is too complex and two impregnated with empirical arguments to be regarded as a rational justification. It only gives hints at such justifications. A first hint is Bohr's correspondence principle, whose historical success suggests that the asymptotic agreement of classical and quantum theory should be a good guide to imagine reasonable axioms for quantum mechanics.

Another hint is the general feeling that a melding of continuity and discontinuity, properly orchestrated by the correspondence principle, should lead to quantum mechanics.

The first hint found spectacular confirmation in Lichnerowicz's and Gutt's proofs that the phase-space formulation of quantum mechanics is the unique (up to an isomorphism) one-parameter deformation of the Poisson algebra of classical mechanics. This result is purely mathematical. It does not imply that the mathematically generated deformation should be a physical theory. However, this deformation is constructed so as to possess a Lie algebra of infinitesimal evolutions. If we share Poincaré's belief in the synthetic a priori character of transformation groups, this makes the deformation a good candidate for being some sort of dynamics. Compared to other arguments for the necessity of quantum mechanics, the deformation approach has an important advantage: it does not only imply the quantum-mechanical characterization of states and their evolution, it also gives the expression of the Hamiltonian and other observables.

The other hint from history, that quantum mechanics should result from a correspondence-guided melding of continuity and discontinuity, is confirmed by Hardy's axiomatics, which generates the matrix-density representation of physical states and their unitary evolution by postulating discrete measurement outcomes and continuous variations of the kind of measurement. The latter principle of continuity, and most of Hardy's other axioms are implicit consequences of some correspondence between classical and quantum theory. The least convincing of Hardy's axioms, the one requiring the lowest value for the number of degrees of freedom associated to a given dimension, is now known to derive from his other axioms. The theorists who corrected this defect moved toward a fuller information-theoretic expression of the axiom. Although this tendency has the advantage of bringing quantum mechanics closer to its applications to the processing of information, it diminishes the necessity of the axioms by severing them from the correspondence arguments that were available in Hardy's original formulation.

In the popular perception of quantum mechanics, its main peculiarities are quantum discontinuity (discrete character of physical quantities that used to be continuous), the existence of incompatible measurements (uncertainty relations), and the existence of entangled states (in Schrödinger's sense). Hardy's axiomatics postulates quantum discontinuity; and the information-theoretic axiomatics of Chiribella, d'Ariano, and Perinotti postulates entangled states through their principle of purification. A much earlier kind of axiomatics, the quantum logic initiated by Neumann and Birkhoff, begins with incompatible measurements. Its focus on experimental Yes-No questions makes it an impoverished logic based on a non-distributive lattice from a formal point of view. Its elegance lies in the economy of its presuppositions and in the power of the mathematics deployed to derive a generalized Hilbert-space representation of the lattice of propositions. Its main defects, compared to Hardy's approach, are the required level of mathematical competence and the failure to single out the Hilbert-space representation among all representations compatible with the basic lattice structure. The axioms of quantum logic are most natural in the finite-dimensional case, and a bit contrived in the infinite-dimensional case developed by Piron and others. Once completed with a definition of states through the statistics of binary tests, they lead to the matrix-density representation of states and to their unitary evolution if the Hilbert-space representation of the logical lattice is selected.

In order to appreciate the kind of necessity of the various axiomatics encountered in this essay, one must examine the nature of the primitive notions needed to formulate the axioms. In the case of quantum logic, the basis notion is that of a repeatable binary test (Piron's "measurements of the first kind"). The theory is constructed from this highly idealized notion, without any information regarding the concrete realization of the tests. There is little doubt, however, that the existence of such tests is a minimal requirement about the possibility of experimentation: We must somehow be able to determine the properties of a system through reliable tests, and every test is evidently traceable to a set of binary tests; a clear cut answer to the latter kind of test is a plausible idealization. As long as the axioms regarding the combination of binary tests are physically reasonable, their consequences seem quite necessary. This extreme generality is the main attraction of quantum logic. It also is its main deficiency. Even after being completed by a statistical definition of states in Mackey's manner, the theory does not give any concrete instructions about how to perform tests and measurements. It only suggests that single measurements should correspond to mutually compatible tests that generate a decomposition of the identity as a sum of orthogonal projectors. In order to associate a measurement value to a given projector, further considerations are needed, perhaps symmetry considerations (in particular, the Hamiltonian is an observable related to the uniformity of time) or correspondence arguments. So to say, quantum logic is an empty shell waiting for an imbedded theory of measurement.[97]

This state of affairs brings out an unusual aspect of the relation between mathematics and physics. According to Helmholtz and Poincaré, the possibility of measurement (in a strong sense including the concrete addition of quantities) is responsible for much of the mathematical structure found to be necessary in formulating physical theories. In quantum logic, the basic notions do not include measurement; they only include binary tests. Interestingly, the specific lattice structure of these tests implies much mathematics, including generalized Hilbert spaces, which are much more advanced mathematical constructs than the real numbers associated with ideal measurement. This mathematics does not imply numbers determinable by experiment. Such numbers only occur at the probabilistic stage at which the statistical correlation between successive tests is defined. Measurement *stricto sensu* remains irrelevant until the quantum logic shell is filled with appropriate metric notions. In a neo-Kantian reading, quantum logic may be seen as a very basic precondition of experiment, prior to the measurability conditions expressed in Helmholtz's doctrine.

At first glance, Hardy's axiomatics does not seem to share this pre-metric quality of quantum logic, since it presupposes "single shot measurements" with definite (discrete) numerical outcomes. In fact, the basic notion of this theory is that of states defined through the probabilities of the various measurement outcomes. This difference with quantum logic is not so great, however, because Hardy's measurements are only defined *in abstracto*, without any prescription for their concrete realization. A more significant difference stems from Hardy's introducing the statistical concept of state at the very beginning of his theory. This difference implies a different sort of necessity for the axioms in the two approaches. Whereas in quantum logic necessity is inherent in the logic of tests, in Hardy's theory it derives from the correspondence arguments that implicit sustain the axioms. As the latter kind of necessity seems vaguer than the former,

[97] Jauch was well aware of this shortcoming and did his best to remedy it in Jauch 1968.

one may be tempted to conclude that quantum logic, despite its more difficult mathematics and its less determined conclusions, better shows the necessity of quantum mechanics than axiomatics à la Hardy. This would be a hurried conclusion, however, because both approaches in the end need correspondence arguments to concretely determine what should be measured on quantum systems.

In this essay, I have given much attention to the degree in which various developments can pass for rational derivations of quantum mechanics. This does not mean, however, that the actors of these developments truly had rationalist ambitions. Birkhoff and Neumann define the aim of their foundational paper as "to discover what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic." Although they occasionally ask for "a plausible physical motivation" for their axioms, they seem to be more concerned with mathematical fertility. Mackey's main concern clearly is clean axiomatization in Hilbert's sense. Other authors profess an operational approach in which the deployed mathematics should be mostly dictated by idealized operations. Yet they do no place the necessity of the operational axioms at the top of their agenda. Piron and Jauch purport to define the most adequate language of quantum theory. They spend relatively little time justifying their axioms. Hardy and his followers insist on the "natural" or "reasonable" character of their axioms in the context of probability theory of information theory. Although they probably mean this naturalness to imply a kind of necessity, it should be distinguished from applicability to physical systems.[98]

To sum up, we have three sorts of arguments for the necessity of quantum mechanics: historical, mathematical, and operational. The historical ones are too impregnated with empirical knowledge to be regarded as rational; the mathematical ones are purely rational but they leave the physical significance of the deduced theory open; the operational ones (quantum logic, or probabilistic states) come closest to rational deductions of quantum mechanics, although their inventors prudently avoided to claim so much, and although some of the basic assumptions, especially the discreteness of measurement outcomes and the existence of incompatible measurements remain largely empirical. With this concession, one cannot help being impressed by the fact that these assumptions, together with basic preconditions of experience and correspondence arguments, lead to the Hilbert-space formalism of quantum mechanics. We thus understand why microphysics needs a kind of mathematics earlier believed to belong to the pure mathematician. We also become convinced that quantum mechanics is the only plausible generalization of classical mechanics that takes into account the basic atomicity of physical phenomena.

[98] Birkhoff and Neumann 1936, pp. 823, 837; Hardy 2001.